ON THE DISTRIBUTION OF VOLUME IN CONVEX BODIES

Thesis for the degree of
Doctor of Philosophy

By
Emanuel Milman

Submitted to the Scientific Council of
the Weizmann Institute of Science
Rehovot, Israel

Prepared at the Mathematics Department
under the supervision of
Professor Gideon Schechtman

JUNE 2007
To my wife, Ronit
Acknowledgements

First and foremost, I would like to express my deepest gratitude to my supervisor, Professor Gideon Schechtman, for many informative discussions, for always carefully reading my texts, and especially for believing in me and allowing me to pursue my interests. I regret that we did not get the chance to work on some problem together, an experience which I hope will come our way in the future.

I would also like to thank my co-authors, Bo’az Klartag and Sasha Sodin. I had a good time working with them, and learned a lot from them both. I have a lot of respect for these two prominent Mathematicians, and I hope we will have a chance to collaborate again in the near future. Extra thanks to Bo’az for his help with my Princeton post-doc application.

Next, I would like to thank Professor Alexander Koldobsky. I enjoyed studying his works, and I am very happy to have answered one of his questions (although not in the direction I had hoped). Special thanks are due for the invitation to visit the University of Missouri, which I also enjoyed very much, and for the letters of reference.

Many thanks go out to Professor Erwin Lutwak for useful advice on submissions of papers and post-doc applications, as well as for the letters of reference. I look forward to meeting him in person sometime soon.

I would also like to thank Professor Apostolos Giannopoulos for being a great Chairman, delivering many excellent talks and of course his letters of reference.

I would like to thank Professor Itai Benjamini for his pep-talks and for being a cool guy and friend. I wish I could have learned more Geometry and Harmonic Analysis from him.

Almost last but not least, I would like to thank my father, Professor Vitali Milman, the relation to whom I am usually in a habit of hiding, but not on this occasion. Apart from setting the foundations to the theory which I am trying to devote at least the next few years to, I am also thankful for his advice concerning Mathematical life in general.

On the same note, thank you Mama for being your wonderful self.

And dearest to my heart, thank you Ronit for all your support throughout the years, and for agreeing to be the wife of a Mathematician.

Emanuel Milman,
June 2007
Table of Contents

Abstract 1

Introduction 2

1 Generalized Intersection Bodies
   *Journal of Functional Analysis 240 (2), 530-567, 2006* 15

2 Generalized Intersection Bodies are not equivalent
   *To appear in Advances in Mathematics* 50

3 A comment on the low-dimensional Busemann-Petty problem
   *Lecture Notes in Math. 1910, GAFA Seminar Notes 2004-5, 245-253* 66

4 Dual Mixed Volumes and the Slicing Problem
   *Advances in Mathematics 207 (2), 566-598, 2006* 74

5 On volume distribution in 2-convex bodies
   *with Bo’az Klartag, to appear in Israel Journal of Mathematics* 104

6 On Gaussian marginals of uniformly convex bodies
   *Submitted* 126

7 Isoperimetric inequalities for uniformly log-concave measures and uniformly convex bodies
   *with Sasha Sodin, Submitted* 146

8 A remark on two duality relations
   *Integral Equations and Operator Theory 57 (2), 217-228, 2007* 174

Abstract in Hebrew 184
ABSTRACT

This thesis has its origins set in the study of several fundamental questions on the asymptotic distribution of volume inside convex bodies in $\mathbb{R}^n$, when $n$ tends to infinity. First, we consider the low-dimensional Busemann-Petty problem. We provide a partial positive answer to this problem, extending a result of Bourgain and Zhang. We also consider a natural question posed by Koldobsky on the equivalence of two classes of generalized intersection bodies, a positive answer to which would yield a positive answer to the low-dimensional Busemann-Petty problem. First, we provide strong evidence in favor of a positive answer to Koldobsky’s question, by proving that both classes share many identical structural properties. Then, we construct a surprising counter-example, completely settling Koldobsky’s question in the negative. Several equivalent formulations of this result are obtained, implying in particular the existence of non-trivial positive functions in the range of the Spherical Radon Transform, and of non-trivial homogeneous functions on $\mathbb{R}^n$ which are both positive and positive-definite (as distributions).

Next, we consider the Slicing Problem, originally posed by Bourgain in the 1980’s, and which is still open for general convex bodies. We partially recover, strengthen and unify into a single framework some results of Ball and Junge on the isotropic constant of unit-balls of subspaces and quotients of $L_p$ ($1 < p < \infty$), and extend these results to negative values of $p$ using generalized intersection bodies. In a joint work with Bo’az Klartag, we study the isotropic constant and other volumetric properties of uniformly convex bodies having power type 2. In the same work, we also obtain a positive answer to the Central Limit Problem for these bodies, a question originally posed by Antilla-Ball-Perissinaki and Brehm-Voigt.

Continuing the study of uniformly convex bodies, we show the existence of approximately Gaussian marginals for several rich sub-families of uniformly convex bodies, including all subspaces of quotients of $L_p$ ($1 < p < \infty$), and all uniformly convex bodies of power type $p$, for $2 \leq p < 4$.

In a joint work with Sasha Sodin, we obtain isoperimetric inequalities for uniformly convex bodies, and for uniformly log-concave measures, a novel notion which we introduce. Our inequalities essentially recover the classical Gromov-Milman concentration for uniformly convex bodies, the isoperimetric inequality for the Gaussian measure of Sudakov-Tsirel’son and Borell (and a generalization of it due to Bakry-Ledoux), an isoperimetric inequality of Bobkov-Zegarlinski, and a log-Sobolev type inequality of Bobkov-Ledoux.

We conclude by combining two known results on the duality of entropy (or covering) numbers due to Bourgain-Pajor-Szarek-Tomeczak-Jaegermann and Artstein-Szarek-Milman, thereby improving, from polynomial to logarithmic in the dimension, the best known bounds on the duality of entropy for general symmetric convex bodies. We use this result to prove a duality relation for Talagrand’s $\gamma_p$ functionals.
INTRODUCTION

The systematic study of volumetric properties of convex bodies in finite dimensional vector space, probably has its origins set in the pioneering works of H. Brunn and H. Minkowski in the late 19th century. Their work was continued by other Geometrists such as Fenchel, Alexandrov, Blaschke, Santalo, Urysohn, Busemann, Petty, Rogers and others. We refer to the books [27] and [52] for an Historical and Mathematical account. In the meanwhile, modern Functional Analysis, aimed at the study of infinite dimensional Banach spaces, was taking shape and form. Apart from the obvious observation that the unit ball of a Banach space is a convex set, there seemed to be little interaction between these two fields. It was only in the late 1960’s and 1970’s, following the celebrated works of A. Grothendieck and A. Dvoretzky, that it was realized that many questions stemming from Functional Analysis are in fact asymptotic questions about finite dimensional Banach spaces, or equivalently, about (centrally symmetric) finite dimensional convex bodies, when the dimension tends to infinity. Thus the study of the local theory of Banach spaces was initiated. Many new concepts and methods were introduced into this rapidly emerging field. Two of the main examples are the concentration of measure phenomenon, introduced by V. Milman, and the type/cotype theory, whose main contributors were B. Maurey and G. Pisier. The 1986 book by V. Milman and G. Schechtman, entitled “Asymptotic Theory of Finite Dimensional Normed Spaces” [44], is nowadays a classical account of these and other developments. Other important texts include the books [47], [48], [57], and the survey paper [40].

In the following years, it became gradually apparent that many questions about properties of convex bodies are interesting in their own right and have their own analytic and geometric merit, independently of their pertinence to Banach space theory. In addition, not only did Geometry begin to play a role as a tool for proving results, but in fact novel geometric consequences were being discovered. And so, at the intersection of classical Convex Geometry and the local theory of Banach spaces, the study of asymptotic properties of convex bodies in $\mathbb{R}^n$ as $n$ tends to infinity, later dubbed “Asymptotic Convex Analysis”, was initiated. Since its conception, this field has been a source of rich and fruitful research, with connections to many other fields, such as Probability Theory, Harmonic Analysis, Mathematical Physics, Computer Science and Combinatorics. We refer to the books [28, 29] for a broad and comprehensive survey on the Geometry of Banach spaces in general and Asymptotic Convex Analysis in particular (especially Chapters 4,17,30,37 which were most influential to our work).

A central theme underlying many fundamental questions in Asymptotic Convex Analysis concerns the distribution of volume inside convex bodies. As suggested by its title, this thesis addresses several manifestations of such volumetric questions. The eight research papers which constitute the chapters of this thesis, were motivated by roughly five (partly...
Lastly, we let $G$ denote the Lebesgue measure of a set $L$ with non-empty interior in $K$. We denote the Lebesgue measure of a set $L$ in $\mathbb{R}^n$ with non-empty interior in $K$ on a different volumetric property and consequently possesses a different analytic flavor.

We will use the following notations throughout the introduction. A convex, compact set $K$ with non-empty interior in $\mathbb{R}^n$ is called a convex body. $K$ is called centrally-symmetric if $K = -K$, i.e. $x \in K$ iff $-x \in K$. We will usually assume that some Euclidean structure has been fixed on $\mathbb{R}^n$, and denote by $D_n$ the Euclidean unit ball, and by $S^{n-1}$ the Euclidean unit sphere. We denote the Lebesgue measure of a set $L \subset \mathbb{R}^n$ in its affine hull by $\text{Vol}(L)$. Lastly, we let $G(n,k)$ denote the Grassmann manifold of $k$-dimensional subspaces in $\mathbb{R}^n$.

1. The Low-Dimensional Busemann-Petty Problem

The questions studied in the first three chapters of this thesis were motivated by the so-called “low-dimensional Busemann-Petty problem”, which will be described below. Although this problem pertains to convex bodies, we will see that some naturally arising questions will lead us outside the realm of convex bodies and into the more general class of star-bodies. A centrally-symmetric star-body $K$ is a compact set with non-empty interior such that $K = -K$, $tK \subset K$ for all $t \in [0,1]$, and such that its radial function $\rho_K(\theta) = \max\{r \geq 0 \mid r\theta \in K\}$ for $\theta \in S^{n-1}$ is an even continuous function on $S^{n-1}$. The class of star-bodies is clearly closed in the natural radial-metric $d_r$, defined by $d_r(K_1,K_2) = \max_{\theta \in S^{n-1}} |\rho_{K_1}(\theta) - \rho_{K_2}(\theta)|$.

The original Busemann-Petty problem [17], first posed in 1956, asks whether two centrally-symmetric convex bodies $K$ and $L$ in $\mathbb{R}^n$ satisfying:

\begin{equation}
\text{Vol}(K \cap H) \leq \text{Vol}(L \cap H) \quad \forall H \in G(n,n-1)
\end{equation}

necessarily satisfy $\text{Vol}(K) \leq \text{Vol}(L)$. For a long time this was believed to be true (this is certainly true for $n = 2$), until a first counterexample was given in 1975 by Larman and Rogers [39] for large values of $n$. In the same year, the notion of an intersection-body was first introduced by E. Lutwak ([41], [42]) in connection to the Busemann-Petty problem. It was shown in [42] (and refined by R. Gardner in [20]) that the answer to the Busemann-Petty problem in dimension $n$ is equivalent to whether all centrally-symmetric convex bodies in $\mathbb{R}^n$ are intersection bodies. Subsequently, it was shown in a series of results ([39], [7], [12], [23], [45], [20], [21], [35], [60], [22]), that this is true for $n \leq 4$, but false for $n \geq 5$.

In [59], G. Zhang considered a natural generalization of the Busemann-Petty problem, which asks whether two centrally-symmetric convex bodies $K$ and $L$ in $\mathbb{R}^n$ satisfying:

\begin{equation}
\text{Vol}(K \cap H) \leq \text{Vol}(L \cap H) \quad \forall H \in G(n,n-k)
\end{equation}

for some integer $k$ between 1 and $n-1$, necessarily satisfy $\text{Vol}(K) \leq \text{Vol}(L)$. Zhang showed that the generalized $k$-codimensional BP-problem is also naturally associated to a notion generalizing that of an intersection-body, which we will refer to as a $k$-Busemann-Petty body. We will denote the class of all $k$-Busemann-Petty bodies in $\mathbb{R}^n$ by $\mathcal{BP}_k^n$ (note that these bodies are referred to as $n-k$-intersection bodies in [59] and generalized $k$-intersection bodies in [38]). In analogy to the original Busemann-Petty problem (the $k = 1$ case), Zhang showed that the generalized $k$-codimensional problem is equivalent to whether all convex bodies in $\mathbb{R}^n$ are $k$-Busemann-Petty bodies. Subsequently, it was shown by J. Bourgain
and Zhang in [15] (see also [50]), and later by A. Koldobsky in [38], that the answer to the
generalized \( k \)-codimensional problem is negative for \( k < n - 3 \), but the cases \( k = n - 3 \) and
\( k = n - 2 \) (the so-called low-dimensional BP-problem) remain open in general. Note that the
case \( k = n - 1 \) is obviously true.

In [38], a second generalization of the notion of an intersection body was introduced by
Koldobsky, who studied a different analytic generalization of the Busemann-Petty problem.
We will refer to these bodies as \( k \)-intersection-bodies, and denote the class of all such bodies
in \( \mathbb{R}^n \) by \( \mathcal{I}_k^n \). We refer to Chapter 1 for complete definitions of the various classes mentioned
above, and emphasize that \( \mathcal{BP}_1^n \) and \( \mathcal{I}_1^n \) are both exactly the class of intersection-bodies in
\( \mathbb{R}^n \).

Koldobsky considered the relationship between these two types of generalizations, \( \mathcal{BP}_k^n \)
and \( \mathcal{I}_k^n \), and proved that \( \mathcal{BP}_k^n \subset \mathcal{I}_k^n \) (see Chapter 1 for a more direct argument). Koldobsky
also asked whether the opposite inclusion is equally true for all \( k \) between 2 and \( n - 2 \) (for
1 and \( n - 1 \) this is true):

**Question (Koldobsky):** Is it true that \( \mathcal{BP}_k^n \subset \mathcal{I}_k^n \) for \( n \geq 4 \) and \( 2 \leq k \leq n - 2 \)?

If this were true, as remarked by Koldobsky, a positive answer would follow to the low-
dimensional BP-problem, i.e. the generalized \( k \)-codimensional BP-problem for \( k \geq n - 3 \),
since for those values of \( k \) any centrally-symmetric convex body in \( \mathbb{R}^n \) is known to be a
\( k \)-intersection body ([36],[37],[38]).

Motivated by the possible connection between Koldobsky’s question and the study of vol-
umetric properties of convex bodies, Chapters 1 and 2 are devoted to describing a complete
solution to this question.

In Chapter 1, we show that the classes \( \mathcal{BP}_k^n \) and \( \mathcal{I}_k^n \) share many identical structural
properties, suggesting that it is indeed reasonable to believe that \( \mathcal{BP}_k^n \subset \mathcal{I}_k^n \). The class \( \mathcal{BP}_k^n \)
is defined using the Integral Geometry language of Spherical Radon Transforms, whereas
the class \( \mathcal{I}_k^n \) is defined using the analytical language of Fourier transforms of homogeneous
distributions. Consequently, we use completely different tools to prove the exact same
results for each class. We define the \( k \)-radial sum of two star-bodies \( L_1, L_2 \) as the star-body
\( L \) satisfying \( \rho_L^k = \rho_{L_1}^k + \rho_{L_2}^k \).

**Structure Theorem.** Let \( \mathcal{C} = \mathcal{I} \) or \( \mathcal{C} = \mathcal{BP} \) and \( k, l = 1, \ldots, n - 1 \). Then:

1. \( \mathcal{C}_k^n \) is closed under full-rank linear transformations, \( k \)-radial sums and taking limit
   in the radial metric.
2. \( \mathcal{C}_1^n \) is the class of intersection-bodies in \( \mathbb{R}^n \), and \( \mathcal{C}_{n-1}^n \) is the class of all symmetric
   star-bodies in \( \mathbb{R}^n \).
3. Let \( K_1 \in \mathcal{C}_{k_1}^n \), \( K_2 \in \mathcal{C}_{k_2}^n \) and \( l = k_1 + k_2 \leq n - 1 \). Then the star-body \( L \) defined by
   \( \rho_L^l = \rho_{K_1}^{k_1} \rho_{K_2}^{k_2} \) satisfies \( L \in \mathcal{C}_l^n \). As corollaries:
   (a) \( \mathcal{C}_{k_1}^n \cap \mathcal{C}_{k_2}^n \subset \mathcal{C}_{k_1 + k_2}^n \) if \( k_1 + k_2 \leq n - 1 \).
   (b) \( \mathcal{C}_k^n \subset \mathcal{C}_l^n \) if \( k \) divides \( l \).
   (c) If \( K \in \mathcal{C}_{k/l}^n \) then the star-body \( L \) defined by \( \rho_L = \rho_K^{k/l} \) satisfies \( L \in \mathcal{C}_l^n \) for \( l \geq k \).
4. If \( K \in \mathcal{C}_k^n \) then any \( m \)-dimensional central section \( L \) of \( K \) (for \( m > k \)) satisfies
   \( L \in \mathcal{C}_m^n \).
(1) and (2) above are well known and basically follow from the definitions, but we mention them here for completeness. (3) for $T_k^n$ was also noticed independently by Koldobsky, but never published. For $BP_k^n$, (4) and (3b) for $k = 1$ were proved by Grinberg and Zhang in [25].

Using a Functional Analytic approach, we also provide in Chapter 1 several surprising equivalent formulations to Koldobsky’s question, which reveal a deep connection to several fundamental problems in the Integral Geometry of the Grassmann Manifold. Let $R_m$ denote the $m$-dimensional Spherical Radon Transform, given by:

$$R_m : C(S^{n-1}) \rightarrow C(G(n, m)) \quad R_m(f)(E) = \int_{S^{n-1} \cap E} f(\theta)d\sigma_E(\theta),$$

where $\sigma_E$ is the Haar probability measure on $S^{n-1} \cap E$. Let $C_+(S^{n-1})$ denote the cone of non-negative continuous functions on the sphere, and let $R_n-k(C(S^{n-1}))_+$ denote the cone of non-negative functions in the image of $R_{n-k}$. Let $I : C(G(n, k)) \rightarrow C(G(n, n-k))$ denote the operator defined as $I(f)(E) = f(E^\perp)$, and let $\overline{A}$ denote the closure of a set $A$ in the corresponding normed space. In particular, we show in Chapter 1 that $BP_k^n = T_k^n$ iff:

$$(1.3) \quad R_{n-k}(C(S^{n-1}))_+ = R_{n-k}(C_+(S^{n-1})) + I \circ R_k(C_+(S^{n-1})).$$

We verify in Chapter 1 that the right-hand side above is contained in the left-hand one, so the problem is equivalent to showing that a non-negative function in the image of $R_{n-k}$ can (essentially) only be obtained in two “trivial” manners. We also show that $BP_k^n = T_k^n$ iff $BP_{n-k}^n = T_{n-k}^n$, revealing again the symmetry between the $k$ and $n-k$ cases.

Despite this and other evidence in Chapter 1 for a positive answer to Koldobsky’s question, we construct the following surprising counter-example in Chapter 2, by analyzing the action of the Spherical Radon Transform on radial functions of bodies of revolution. Let $O(n)$ denote the orthogonal group on $\mathbb{R}^n$. Recall that a star-body $K$ is called a body of revolution if its radial function $\rho_K \in C(S^{n-1})$ is invariant under the natural action of $O(n-1)$ identified as some subgroup of $O(n)$.

**Negative Solution to Koldobsky’s Question.** Let $n \geq 4$ and $2 \leq k \leq n-2$. Then there exists an infinitely smooth centrally-symmetric body of revolution $K$ such that $K \in T_k^n$ but $K \notin BP_k^n$.

We remark that the $K$ we construct cannot be a convex body (see Chapter 2), so this result does not imply a negative answer to the unresolved cases $k = n-2, n-3$ (for $n \geq 5$) of the low-dimensional BP-problem. We conclude that in any attempt to prove a positive answer to these unresolved cases by means of comparing $k$-intersection bodies to $k$-Busemann-Petty bodies, it is essential to restrict one’s attention to convex bodies.

Our negative solution implies the negation of the statement in (1.3). Since this statement is obtained using cone-duality and the Hahn-Banach Theorem, we can only infer the non-constructive existence of an (infinitely smooth) function $f \in R_{n-k}(C(S^{n-1}))_+$ which cannot be approximated (in $C(G(n, n-k))$) by functions of the form $R_{n-k}(g) + I \circ R_k(h)$ with $g, h \in C_+(S^{n-1})$.

Other equivalent negative formulations using the language of Fourier transforms of homogeneous distributions are given in Chapter 2. We comment here that one such formulation pertains to embeddings in $L_p$ for negative values of $p$. The definition of this notion of
embedding (for \(-n < p < 0\)) was given by Koldobsky in [38] by means of analytic continuation of one of the equivalent definitions for \(p > 0\). It is known that for \(p \geq -1\) \((p \neq 0)\) and for \(-n < p \leq -n + 1\), any star-body \(K\) such that \((\mathbb{R}^n, \|\cdot\|_K)\) embeds in \(L_p\) can be generated in a “trivial” manner, by starting with the Euclidean ball \(D_n\), applying full-rank linear transformations, \((-p)\)-radial sums and taking the limit in the radial metric. Using a characterization of the class \(\mathcal{BP}_n^k\) due to Grinberg and Zhang [25], our results imply that \(p = -1\) and \(p = -n + 1\) are critical values for this property, and that this is no longer true for \(p = -k\), \(2 \leq k \leq n - 2\). Alternatively, our counter-example can be equivalently restated as a construction of a non-trivial homogeneous (infinitely smooth) function on \(\mathbb{R}^n\) which is both positive and positive-definite (as a distribution); we refer to Chapter 2 for more details.

Chapter 3 concerns a partial positive answer to the low-dimensional BP-problem. Several partial answers to the unresolved cases in (1.2) were previously known. It was shown by Zhang in [59] (see also [50]) that when \(K\) is a centrally-symmetric convex body of revolution then the answer is positive for the pair \(K, L\) with \(k = n - 2, n - 3\) and any star-body \(L\). When \(k = n - 2\), it was shown by Bourgain and Zhang in [15] that the answer is positive if \(L\) is a Euclidean ball and \(K\) is convex and sufficiently close to \(L\). In Chapter 3, we extend this result by showing that the answer is positive in the cases \(k = n - 2, n - 3\) when \(L\) is an arbitrary body and \(K\)’s radial function is an \(k\)-th root of the radial function of a convex body. In particular, this implies that when \(K\) is sufficiently close to a Euclidean ball (to an extent depending on its curvature), the answer is positive for any star-body \(L\). Recently, B. Rubin [49] generalized Zhang’s result by showing that a positive answer holds whenever \(K\) is a convex body with certain rotational axial symmetries.

2. The Slicing Problem

One of the most famous and intriguing problems in Asymptotic Convex Analysis is the so called Slicing Problem (or Hyperplane Conjecture), originally posed by J. Bourgain in the 1980’s. It asks whether any (centrally-symmetric) convex body \(K\) in \(\mathbb{R}^n\) of volume 1, has an \(n - 1\) dimensional central section whose volume is bounded from below by a universal constant \(c > 0\) not depending on \(K\) or \(n\).

This problem is known to have many equivalent formulations (c.f. V. Milman and A. Pajor [43]), which basically follow by properly applying some variant of the classical Brunn-Minkowski inequality. For instance, it is easy to see that any convex body \(K\) in \(\mathbb{R}^n\) has an affine image of volume 1 such that the uniform distribution on \(K\) has covariance matrix \(L_K^2 I_n\), where \(I_n\) is the identity \(n\) by \(n\) matrix and \(L_K\) is some positive constant. This image, which is unique modulo rotations, is called the isotropic position of \(K\), and \(L_K\) is called the isotropic constant of \(K\). Then the Slicing Problem is equivalent to asking whether for any convex body \(K\) in \(\mathbb{R}^n\), \(L_K\) is bounded from above by a universal constant independent of \(K\) or \(n\). Alternatively, it is easy to show that \(L_K^2 = \frac{1}{n} \inf_{K'} \int_{K'} |x|^2 dx\) where the infimum is over all affine images \(K'\) of \(K\) of volume 1. Using this, it is elementary to show that the Slicing Problem is equivalent to asking whether for any convex body \(K\) there exists an ellipsoid \(E\) such that \(\text{Vol}(E \cap K) \geq \text{Vol}(K) / 2\) and \((\text{Vol}(E) / \text{Vol}(K))^{1/n} \) is bounded from above by a universal constant, independent of \(K\) or \(n\).
Finally, we mention that the Slicing Problem is also equivalent to a variant of the Busemann-Petty problem, where one wishes to conclude from (1.1) that $\text{Vol}(K) \leq C\text{Vol}(L)$, for some universal constant $C > 1$ independent of $n$. The negative solution to the BP-problem implies that we cannot ask for $C = 1$ when $n \geq 5$, but it is still plausible that a larger constant may suffice.

The Slicing Problem is known to have a positive answer for many families of convex bodies (see Chapter 4), but the general case is still open. Up until recently, the best general bound on $L_K$ as a function of $n$ for an arbitrary convex body in $\mathbb{R}^n$ was due to Bourgain [13], who showed that $L_K \leq Cn^{1/4}\log(n)$, for some constant $C > 0$. This was recently improved by B. Klartag [32] to $L_K \leq Cn^{1/4}$. We refer to the excellent survey [24] by A. Giannopoulos for further details on isotropic convex bodies.

In Chapter 4, we develop a technique for bounding the isotropic constant of $K$, by comparing $K$ with a (non necessarily convex) body $L$ containing $K$, which is chosen from a less general family of bodies. We consider two main families: unit-balls of $n$-dimensional subspaces and quotients of $L_p$, denoted $SL^n_p$ and $QL^n_p$, respectively, and the already familiar $k$-Busemann-Petty bodies, denoted $BP^n_k$. Our main tool for comparing two star-bodies is the notion of dual mixed-volumes, first introduced by E. Lutwak in [41] as the dual counterpart to the classical mixed-volumes of the Brunn-Minkowski theory (see Chapter 4).

In particular, our approach partially recovers and strengthens results of K. Ball [6] and M. Junge [30] concerning the isotropic constants of bodies in $SL^n_p$ and $QL^n_p$. Ball showed that the members of $SL^n_p$ for the range $1 \leq p \leq 2$ (in fact it is known that these are all subsets of the class $SL^n_1$) have a universally bounded isotropic constant. Junge’s results apply to $SQL^n_p$, the class of $n$-dimensional unit balls of subspaces of quotients of $L_p$, for $1 < p < \infty$. In particular, his estimates provide the best known bounds on the isotropic constant of members in $SL^n_p$ for $2 \leq p < \infty$ and $QL^n_p$ for $1 < p \leq 2$. The classes $QL^n_p$ for $2 \leq p \leq \infty$ are all subsets of $QL^n_{\infty}$, also known as the class of zonoids or projection bodies. The latter class is known to have a uniformly bounded isotropic constant as well. Our approach provides a single framework to handle the entire range $1 \leq p < \infty$ for $SL^n_p$ and $1 < p \leq \infty$ for $QL^n_p$, and our estimates provide somewhat more information on the isotropic constant. The main advantage of our methods is that they do not rely on heavy tools from the local theory of Banach spaces, as do the other approaches, thereby retaining the Slicing Problem’s geometric flavor. We also generalize the results for $SL^n_p$ to values of $p$ less than 1, and obtain a bound on the isotropic constant of convex bodies in $BP^n_k$ and their duals.

In Chapter 5, which is a joint work with Bo’az Klartag, we concentrate on another class of convex bodies: uniformly convex bodies of power type 2, or 2-convex bodies for short. The modulus of convexity of $K$ is defined as the following (affine invariant) function for $0 < \varepsilon \leq 2$:

$$\delta_K(\varepsilon) = \inf \left\{ 1 - \frac{\|x + y\|_K}{2} : \|x\|_K, \|y\|_K \leq 1, \|x - y\|_K \geq \varepsilon \right\},$$

where $\|\cdot\|_K$ is the norm associated with $K$. $K$ is called uniformly convex if $\delta_K(\varepsilon) > 0$ for all $0 < \varepsilon \leq 2$, meaning that every boundary point is extremal. $K$ is called $p$-convex (for $p \geq 2$), if for some $\alpha > 0$ and all $0 < \varepsilon \leq 2$, $\delta_K(\varepsilon) \geq \alpha \varepsilon^p$. It is known that the unit balls
of $l_p$, and more generally, members of $SQL_p^n$, for $1 < p < \infty$, are max$(2, p)$-convex (see Chapter 6 and the references therein).

In Chapter 5, we study several aspects of the distribution of volume inside 2-convex bodies and in particular give a simple argument for a positive answer to the Slicing Problem for this class, recovering a previous result of Schmuckenschläger [51]. Our argument relies on a classical concentration of volume inequality for uniformly convex bodies due to M. Gromov and V. Milman [26], which will be described in Section 4. Several interesting properties of these bodies are established, which in particular imply that certain canonical affine images of these bodies are essentially equivalent. Using these techniques, we recover the aforementioned results of Junge about $SQL_p^n$, and in fact improve the best known bounds on the isotropic constant for the range $1 < p < 2$. We also show that for any 2-convex body, there exists a one dimensional marginal of the uniform distribution in $K$ which has approximately Gaussian distribution, improving a similar result by Antilla, Ball and Perissinaki. This is a particular case of the Central-Limit Problem for convex bodies, described next.

3. The Central-Limit Problem for Convex Bodies

The Central-Limit Problem for convex bodies asks whether for every convex body $K$ of volume 1 in $\mathbb{R}^n$, there exists a direction such that the one-dimensional projection (marginal) of the uniform distribution in $K$ onto that direction has approximately Gaussian distribution, with an increasing level of approximation as $n$ tends to infinity. This was conjectured by Antilla, Ball and Perissinaki [1] and Brehm and Voigt [16] (in fact in stronger and more concrete formulations). This problem is a generalization of the classical Central-Limit law to the case where the $n$ random-variables are not independent, but rather represent an $n$-dimensional vector uniformly distributed inside a convex body. Of course this cannot hold for the projection onto any direction as witnessed by the $n$-dimensional cube, whose marginals in the directions of the axes are always uniform distributions. However, the classical Central-Limit Theorem guarantees that the marginal of the cube in the direction of the diagonal $(1/\sqrt{n}, \ldots, 1/\sqrt{n})$ does tend in distribution to an appropriate Gaussian.

In the broader context of general measures on $\mathbb{R}^n$ with finite second moment, Sudakov [53] showed that most marginals are approximately the same mixture of Gaussian distributions. Under additional conditions on the covariance matrix of the measure in question, Diaconis and Freedman [18] showed that this mixture can be replaced by a proper Gaussian. A generalized version of both results was given by von Weizsäcker in [58]. When $K$ is a volume 1 homothetic copy of the Euclidean ball $D_n$, the fact that (all) marginals are approximately Gaussian is classical, dating back to Maxwell, Poincaré and Borel (see [19] for a historical account). Other concrete bodies, such as the cross-polytope and simplex, were studied by Brehm and Voigt in [16]. Antilla, Ball and Perissinaki [1] showed that a positive answer to the Central-Limit Problem for convex bodies would follow from a non-trivial estimate on the level of concentration of the uniform distribution in an isotropic convex body $K$ around a thin spherical shell. In fact, they conjectured that for some $\varepsilon_n \searrow 0$ as $n$ tends to infinity and any isotropic convex body $K$ in $\mathbb{R}^n$:

$$\text{Vol}\left(x \in K; \left| \frac{|x|}{\sqrt{n}L_K} - 1 \right| \geq \varepsilon_n \right) \leq \varepsilon_n.$$
They proved this conjecture for all unit-balls of $l_p$ ($1 \leq p \leq \infty$), and for uniformly convex bodies with some restrictions on their diameter in isotropic position.

In Chapter 6, we extend and strengthen the results from Chapter 5 on the existence of Gaussian marginals to more general convex bodies. We show the existence of marginals which are approximately Gaussian in a very strong metric for $p$-convex bodies with $2 \leq p < 4$. The methods involved include the use of a delicate log-Sobolev type inequality of S. Bobkov and M. Ledoux [9] and an application of M. Talagrand’s “Majorizing Measures Theorem” [56]. Under some additional assumptions, we also extend this to values of $p$ greater than 4, showing in particular that all members of $SQL^n_p$ for $1 < p < \infty$ have Gaussian marginals (in the appropriate sense). Surprisingly, we obtain our results by putting the bodies in non-isotropic positions, to which end we prove a non-isotropic version of the aforementioned result of Antilla, Ball and Perissinaki.

Immediately after the work on Chapter 6 was completed, Bo’az Klartag obtained a positive answer to the Central-Limit Problem for general convex bodies [33]. Klartag’s result is in fact applicable to all log-concave absolutely continuous probability measures in $\mathbb{R}^n$ (see Section 4 for definitions). In addition, for suitable $k$ increasing with $n$, the existence of $k$-dimensional marginals which are approximately Gaussian was also shown. We remark that Klartag’s estimates yield a very slow logarithmic rate (in the dimension $n$) of convergence to the Gaussian law, which are inferior to the polynomial rates we obtain for uniformly convex bodies. In a very recent progress in this direction, Klartag has improved his quantitative estimates for general convex bodies to polynomial ones as well [34].

4. Isoperimetric Inequalities for Uniformly Convex Bodies

Isoperimetric inequalities are a way to describe certain relationships between a metric space $(M, d)$ and a Borel measure $\mu$ on $M$. Recall that the Minkowski (exterior) boundary measure of a Borel set $A$ is defined as:

$$\mu^+_d(A) := \liminf_{\varepsilon \to 0} \frac{\mu(A_{\varepsilon,d}) - \mu(A)}{\varepsilon},$$

where $A_{\varepsilon,d} := \{x \in M; d(x, A) < \varepsilon\}$ is the $\varepsilon$-extension of the set $A$ with respect to $d$. The classical isoperimetric inequality in Euclidean space $(\mathbb{R}^n, |\cdot|)$ with respect to the Lebesgue measure $\lambda$ states that the Euclidean ball has minimal surface-area among all (Borel) sets of given volume. Using our notations, this can be equivalently restated as:

$$\lambda^+_1(A) \geq nw_n^{\frac{1}{n}} \lambda A^{\frac{n-1}{n}},$$

for any Borel set $A$, where $w_n$ denotes the volume of the Euclidean unit ball $D_n$. In general, an isoperimetric inequality has the form $\mu^+_d(A) \geq I(\mu(A))$, where $I : \mathbb{R}_+ \to \mathbb{R}_+$ is some (continuous) function. Isoperimetric inequalities constitute one of the strongest forms of the concentration of measure phenomenon, which roughly means that a Lipschitz function with respect to $d$ is highly concentrated with respect to $\mu$ around its median (or mean). The concentration of measure phenomenon has played a fundamental role in the last 30 years in the development of many fields, such as the local theory of Banach spaces, Probability Theory, Combinatorics, Harmonic Analysis, as well as Asymptotic Convex Analysis.
When studying volumetric properties of convex bodies, the measure $\mu$ is usually chosen to be $\lambda_K$, the uniform probability distribution inside the convex body $K$ in question. The choice of the metric $d$ on $\mathbb{R}^n$ already depends on the application. For instance, in the Central-Limit Problem, a natural choice for $d$ is the Euclidean metric for which $K$ is isotropic, and (3.1) is an example of a concentration inequality for the function $|x|$ with respect to $\lambda_K$. Another natural choice for $d$ is given by $\|\cdot\|_K$, the norm associated with $K$, for which any concentration inequality is automatically affine invariant. It is in this setup that the classical Gromov-Milman concentration inequality [26] for uniformly convex bodies applies. It states (we use here the formulation due to Arias-de-Reyna, Ball and Villa [2]) that for any Borel set $A$:

\begin{equation}
\lambda_K(A) = \frac{1}{2} \implies \lambda_K(A_{\varepsilon,\|\cdot\|_K}) \geq 1 - 2 \exp(-2n\delta_K(\varepsilon)),
\end{equation}

where $\delta_K$ is the modulus of convexity of $K$ (2.1). As already mentioned, this inequality is used in Chapters 5 and 6 to prove that the Euclidean norm is concentrated in some thin spherical shell (as in (3.1)) for uniformly convex bodies in several non-isotropic positions. We now see that these positions are chosen in a manner such that the transition from the norm $\|\cdot\|_K$ to the Euclidean norm $|\cdot|$ carries minimal penalty.

In Chapter 7, which is a joint work with Sasha Sodin, we obtain an isoperimetric inequality for the space $(\mathbb{R}^n, \|\cdot\|_K)$ with respect to $\lambda_K$, which is essentially an isoperimetric analogue of the Gromov-Milman concentration inequality. We show (see Chapter 7 for a more precise formulation) that:

\begin{equation}
\lambda_K^+(A, \|\cdot\|_K) \geq C(n, \delta_K) \gamma(\log \frac{1}{\lambda_K(A)}),
\end{equation}

where $\lambda_K^+(A) := \min(\lambda_K(A), 1 - \lambda_K(A))$, $\gamma(x) = x/\delta_K^{-1}(x)$, and $C(n, \delta_K)$ is a constant which only depends on $n$ and $\delta_K$. By integrating this infinitesimal inequality, we essentially recover the Gromov-Milman concentration inequality, up to the value of the constant in the exponent of (4.1); we refer to Chapter 7 for more details. For $p$-convex bodies, our isoperimetric inequality also recovers (up to universal constants) a log-Sobolev-type functional inequality due to S. Bobkov and M. Ledoux [9], and a Sobolev-type isoperimetric inequality of Bobkov and Zegarlinski [10] for sets $A$ with $\lambda_K(A) \geq \exp(-n)$.

Our method employs a localization argument already used by Gromov and Milman, which was further developed by Kannan, Lovász and Simonovits [31] and advocated by Bobkov (e.g. [8]). With this approach, we cannot work directly with uniform distributions on uniformly convex bodies, and are forced to pass through an auxiliary construction, given by uniformly log-concave measures. Recall that a log-concave measure $\mu$ on $\mathbb{R}^n$ is an absolutely continuous Borel measure whose density $f$ satisfies that $\log f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ is a concave function. There exists a well known analogy between convex bodies and log-concave measures, which has received renewed attention in the last several years. The basic observation, a consequence of the Brunn-Minkowski inequality, is that the measure projection of the uniform distribution on a convex body is a log-concave measure. Our approach extends the known analogy between convex bodies and log-concave measures by introducing a novel notion of uniformly log-concave measures. For this class of measures, we obtain an isoperimetric inequality similar to (4.2) for a uniformly log-concave measure.
mu_K, from which we deduce (4.2) by constructing a Lipschitz map which pushed forward mu_K onto lambda_K. The latter step is given by a general theorem, extending a result of Bobkov and Ledoux [9] proved for certain specific log-concave measures. Our isoperimetric inequality for uniformly log-concave measures generalizes (up to universal constants) the classical Gaussian isoperimetric inequality of V. Sudakov and B. Tsirel'son [54] and C. Borell [11], and a generalization of this inequality to more general measures due to Bakry and Ledoux [5].

5. Duality of Entropy and Covering Numbers

Another interesting conjecture pertains to the duality of entropy numbers of compact operators acting on Banach spaces. This conjecture, which goes back to Pietsch [46] in the 1970’s, can be equivalently formulated using the notion of covering numbers. Given two centrally symmetric convex bodies K and T, the covering number of K by T, denoted N(K, T), is defined as the minimal number of translates of T needed to cover K. We also denote by L the polar (or dual) body to L, defined as L = {x ∈ R^n; ⟨x, y⟩ ≤ 1 ∀y ∈ L}.

Duality Conjecture for Covering Numbers. Do there exist numerical constants a, b ≥ 1 such that for any dimension n and for any two symmetric convex bodies K, T in R^n:

\begin{equation}
\log N(K, T) \leq b \log N(T^\circ, a^{-1}K^\circ)
\end{equation}

There has been much progress in the study of this conjecture in recent years (especially by S. Artstein, V. Milman, S. Szarek and N. Tomczak-Jaegermann [4], [3]), and it was shown to hold whenever one of the bodies K or T belongs to some rich family of convex bodies (bodies which are unit-balls of spaces with non-trivial type, including all p-convex bodies and their duals).

In Chapter 8, we generalize and strengthen the best known results in this direction, by an easy combination of several previously developed tools from [14] and [4]. In particular, this implies that in R^n, one can always choose:

\begin{equation}
a = C \log(1 + n) \quad b = C \log(1 + n) \log \log(2 + n)
\end{equation}

in (5.1), for some universal constant C > 1, solving the conjecture up to these logarithmic terms. This should be compared with the previously known best estimate (to the best of our knowledge) for general centrally-symmetric convex bodies K, T, given by a = Cn^{1/2} and b = C for some C > 1. The novelty of our result in comparison to the results of [3] lies in the logarithmic dependence of the estimates on a, b in the Banach-Mazur distance between K or T and “well-behaved” bodies.

The above result was motivated by the study of the Slicing Problem, which may be approached using the \gamma_2 functional of M. Talagrand, as will be explained below. Given a metric space (M, d) and p > 0, the \gamma_p functional is defined as:

\begin{equation}
\gamma_p(M, d) := \inf \sup_{x \in M} \sum_{j \geq 0} 2^{j/p}d(x, M_j)
\end{equation}

where the infimum runs over all admissible sets \{M_j\}, meaning that M_j ⊂ M and |M_j| = 2^{2^j}. For two symmetric convex bodies K, T in R^n, let us denote \gamma_p(K, T) := \gamma_p(K, d_T), where d_T is the metric corresponding to the norm induced by T. The \gamma_2(\cdot, D) functional,
when $D$ is an ellipsoid, was introduced to study the boundedness of Gaussian processes. It was shown by Talagrand in his celebrated “Majorizing Measures Theorem” [55] (see also [56]), that in fact $\gamma^2(K, D)$ and $\sup_{x \in K} \langle x, G \rangle$, where $G$ is a Gaussian random variable (with covariance corresponding to $D$ in an appropriate manner), are equivalent to within universal constants. This was later extended to various other classes of stochastic processes, where the naturally arising metric $d$ is not the $l_2$ norm. Our result on Pietsch’s conjecture implies the following duality relation:

$$\gamma_p(K, T) \leq C_p \log(1 + n)^{2+1/p} \log \log(2 + n)^{1/p} \gamma_p(T^\circ, K^\circ),$$

for any $p > 0$, where $C_p > 0$ depends solely on $p$.

The connection to the Slicing Problem is based on the observation (essentially due to Bourgain) that, under some general conditions on a convex $K$ of volume 1, $n L^2_K$ is equivalent (up to $\log(n)$ terms as above) to $\inf \gamma_2(K', \Psi_2(K'))$, where the infimum is taken over all volume preserving linear images $K'$ of $K$, and $\Psi_2(T)$ is some natural convex body associated with $T$. A duality relation for $\gamma_2$ therefore opens the door for progress in the study of the Slicing Problem.

REFERENCES

CHAPTER 1

GENERALIZED INTERSECTION BODIES

EMANUEL MILMAN


ABSTRACT. We study the structures of two types of generalizations of intersection-bodies and the problem of whether they are in fact equivalent. Intersection-bodies were introduced by Lutwak and played a key role in the solution of the Busemann-Petty problem. A natural geometric generalization of this problem considered by Zhang, led him to introduce one type of generalized intersection-bodies. A second type was introduced by Koldobsky, who studied a different analytic generalization of this problem. Koldobsky also studied the connection between these two types of bodies, and noted that an equivalence between these two notions would completely settle the unresolved cases in the generalized Busemann-Petty problem. We show that these classes share many identical structural properties, proving the same results using Integral Geometry techniques for Zhang’s class and Fourier transform techniques for Koldobsky’s class. Using a Functional Analytic approach, we give several surprising equivalent formulations for the equivalence problem, which reveal a deep connection to several fundamental problems in the Integral Geometry of the Grassmann Manifold.

1. Introduction

Let Vol (L) denote the Lebesgue measure of a set L ⊂ R^n in its affine hull, and let G(n, k) denote the Grassmann manifold of k dimensional subspaces of R^n. Let D_n denote the Euclidean unit ball, and S^{n-1} the Euclidean sphere. All of the bodies considered in this note will be assumed to be centrally-symmetric star-bodies, defined by a continuous radial function ρ_K(θ) = max{r > 0 | rθ ∈ K} for θ ∈ S^{n-1} and a star-body K. We shall deal with two generalizations of the notion of an intersection body, first introduced by E. Lutwak in [24] (see also [25]). A star-body K is said to be an intersection body of a star-body L, if ρ_K(θ) = Vol (L ∩ θ⊥) for every θ ∈ S^{n-1}, where θ⊥ is the hyperplane perpendicular to θ. K is said to be an intersection body, if it is the limit in the radial metric d_r of intersection bodies {K_i} of star-bodies {L_i}, where d_r(K_1, K_2) = sup_{θ ∈ S^{n-1}} |ρ_{K_1}(θ) - ρ_{K_2}(θ)|. This is equivalent (e.g. [25], [6]) to ρ_K = R^*(dμ), where μ is a non-negative Borel measure on S^{n-1}, R^* is the dual transform (as in (1.3)) to the Spherical Radon Transform R : C(S^{n-1}) → C(S^{n-1}), which is defined for f ∈ C(S^{n-1}) as:

\begin{equation}
R(f)(θ) = \int_{S^{n-1} \cap θ⊥} f(ξ) dσ_θ(ξ),
\end{equation}

where σ_θ the Haar probability measure on S^{n-1} ∩ θ⊥.

Supported in part by BSF and ISF.
1. GENERALIZED INTERSECTION BODIES

The notion of an intersection body has been shown to be fundamentally connected to the Busemann-Petty Problem (first posed in [5]), which asks whether two centrally-symmetric convex bodies \( K \) and \( L \) in \( \mathbb{R}^n \) satisfying:

\[
(1.2) \quad \text{Vol}(K \cap H) \leq \text{Vol}(L \cap H) \quad \forall H \in G(n, n-1)
\]

necessarily satisfy \( \text{Vol}(K) \leq \text{Vol}(L) \). It was shown in [25], [6] that the answer is equivalent to whether all convex bodies in \( \mathbb{R}^n \) are intersection bodies, and in a series of results ([23], [1], [3], [11], [27], [6], [7], [18], [37], [8]) that this is true for \( n \leq 4 \), but false for \( n \geq 5 \).

In [36], G. Zhang considered a generalization of the Busemann-Petty problem, in which \( G(n, n-1) \) in (1.2) is replaced by \( G(n, n-k) \), where \( k \) is some integer between 1 and \( n-1 \). Zhang showed that the generalized \( k \)-codimensional Busemann-Petty problem is also naturally associated to a generalized class of intersection-bodies, which will be referred to as \( k \)-Busemann-Petty bodies (note that these bodies are referred to as \( n-k \)-intersection bodies in [36] and generalized \( k \)-intersection bodies in [21]), and that the generalized \( k \)-codimensional problem is equivalent to whether all convex bodies in \( \mathbb{R}^n \) are \( k \)-Busemann-Petty bodies. It was shown in [4] (see also a correction in [29]), and later in [21], that the answer is negative for \( k < n-3 \), but the cases \( k = n-3 \) and \( k = n-2 \) still remain open (the case \( k = n-1 \) is obviously true). Several partial answers to these cases are known. It was shown in [36] (see also [29]) that when \( K \) is a centrally-symmetric convex body of revolution then the answer is positive for the pair \( K, L \) with \( k = n-2, n-3 \) and any star-body \( L \). When \( k = n-2 \), it was shown in [4] that the answer is positive if \( L \) is a Euclidean ball and \( K \) is convex and sufficiently close to \( L \). Several other generalizations of the Busemann-Petty problem were treated in [29], [38], [34], [35].

Before defining the class of \( k \)-Busemann-Petty bodies we shall need to introduce the \( m \)-dimensional Spherical Radon Transform, acting on spaces of continuous functions as follows:

\[
R_m : C(S^{n-1}) \rightarrow C(G(n, m))
\]

\[
R_m(f)(E) = \int_{S^{n-1} \cap E} f(\theta)d\sigma_E(\theta),
\]

where \( \sigma_E \) is the Haar probability measure on \( S^{n-1} \cap E \). It is well known (e.g. [17]) that as an operator on \( \text{even} \) continuous functions, \( R_m \) is injective. The dual transform is defined on spaces of \( \text{signed} \) Borel measures \( \mathcal{M} \) by:

\[
(1.3) \quad R^*_m : \mathcal{M}(G(n, m)) \rightarrow \mathcal{M}(S^{n-1})
\]

\[
\int_{S^{n-1}} f R^*_m(d\mu) = \int_{G(n, m)} R_m(f)(\theta)d\mu \quad \forall f \in C(S^{n-1}),
\]

and for a measure \( \mu \) with continuous density \( g \), the transform may be explicitly written in terms of \( g \) (see [36]):

\[
R^*_m g(\theta) = \int_{\theta \in E \in G(n, m)} g(E)d\nu_{m, \theta}(E),
\]

where \( \nu_{m, \theta} \) is the Haar probability measure on the homogeneous space \( \{ E \in G(n, m) \mid \theta \in E \} \).

We shall say that a body \( K \) is a \( k \)-Busemann-Petty body if \( \rho^*_K = R^*_{n-k}(d\mu) \) as measures in \( \mathcal{M}(S^{n-1}) \), where \( \mu \) is a non-negative Borel measure on \( G(n, n-k) \). We shall denote the class of such bodies by \( \mathcal{BP}^n_k \). Choosing \( k = 1 \), for which \( G(n, n-1) \) is isometric to \( S^{n-1}/\mathbb{Z}_2 \),
by mapping $H$ to $S^{n-1} \cap H^\perp$, and noticing that $R$ is equivalent to $R_{n-1}$ under this map, we see that $\mathcal{BP}_k^n$ is exactly the class of intersection bodies.

Another generalization of the notion of an intersection body, which was considered by Koldobsky in [21], is that of a $k$-intersection body. A star-body $K$ is said to be a $k$-intersection body of a star-body $L$, if $\text{Vol}(K \cap H^\perp) = \text{Vol}(L \cap H)$ for every $H \in G(n, n-k)$. $K$ is said to be a $k$-intersection body, if it is the limit in the radial metric of $k$-intersection bodies $\{K_i\}$ of star-bodies $\{L_i\}$. We shall denote the class of such bodies by $\mathcal{I}_k^n$. Again, choosing $k = 1$, we see that $\mathcal{I}_1^n$ is exactly the class of intersection bodies.

In [21], Koldobsky considered the relationship between these two types of generalizations, $\mathcal{BP}_k^n$ and $\mathcal{I}_k^n$, and proved that $\mathcal{BP}_k^n \subset \mathcal{I}_k^n$ (hence our reluctance to use the term "generalized n–k-intersection bodies" for $\mathcal{BP}_k^n$). Koldobsky also asked whether the opposite inclusion is equally true for all $k$ between 2 and $n-2$ (for 1 and $n-1$ this is true). If this were true, as remarked by Koldobsky, a positive answer to the generalized $k$-codimensional Busemann-Petty problem for $k \geq n-3$ would follow, since for those values of $k$ any centrally-symmetric convex body in $\mathbb{R}^n$ is known to be a $k$-intersection body ([19],[20],[21]).

Our first remark in this note is that the two classes $\mathcal{BP}_k^n$ and $\mathcal{I}_k^n$ share many identical structural properties, suggesting that it is indeed reasonable to believe that $\mathcal{BP}_k^n = \mathcal{I}_k^n$. Some previously known characterizations of these classes and associated tools are outlined in Section 2, providing some intuitive motivation and common ground to start from. Some of these previously known results are also given simplified proofs in this section. It turns out that the natural language for handling the class $\mathcal{I}_k^n$ is the language of Fourier Transforms of homogeneous distributions, developed extensively by Koldobsky, while the natural language for the class $\mathcal{BP}_k^n$ is the language of Integral Geometry and Radon Transforms. In Section 3 we show that both classes share a common structure, by proving the same results for $\mathcal{BP}_k^n$ (using Grassmann Geometry techniques) and for $\mathcal{I}_k^n$ (using Fourier Transform techniques). We define the $k$-radial sum of two star-bodies $L_1, L_2$ as the star-body $L$ satisfying $\rho^k_L = \rho^k_{L_1} + \rho^k_{L_2}$. For each of these classes $\mathcal{C}_k^n$, where $C = \mathcal{I}$ or $C = \mathcal{BP}$ and $k, l = 1, \ldots, n-1$, we show the following:

**Structure Theorem.**

1. $\mathcal{C}_k^n$ is closed under full-rank linear transformations, $k$-radial sums and taking limit in the radial metric.
2. $\mathcal{C}_1^n$ is the class of intersection-bodies in $\mathbb{R}^n$, and $\mathcal{C}_{n-1}^n$ is the class of all symmetric star-bodies in $\mathbb{R}^n$.
3. Let $K_1 \in \mathcal{C}_{k_1}^n$, $K_2 \in \mathcal{C}_{k_2}^n$ and $l = k_1 + k_2 \leq n-1$. Then the star-body $L$ defined by $\rho_{L_i}^k = \rho_{K_1}^{k_1} \rho_{K_2}^{k_2}$ satisfies $L \in \mathcal{C}_l^n$. As corollaries:
   a. $\mathcal{C}_{k_1}^n \cap \mathcal{C}_{k_2}^n \subset \mathcal{C}_{k_1+k_2}^n$ if $k_1 + k_2 \leq n-1$.
   b. $\mathcal{C}_k^n \subset \mathcal{C}_l^n$ if $k$ divides $l$.
   c. If $K \in \mathcal{C}_k^n$ then the star-body $L$ defined by $\rho_{L_i}^k = \rho_{K_i}^{k/l}$ satisfies $L \in \mathcal{C}_l^n$ for $l \geq k$.
4. If $K \in \mathcal{C}_k^n$ then any $m$-dimensional central section $L$ of $K$ (for $m > k$) satisfies $L \in \mathcal{C}_{k}^m$.

(1) and (2) above are well known and basically follow from the definitions (or from the characterizations in Section 2), but we mention them here for completeness. It should also be clear that (3) implies the three corollaries following it: (3a) by using $K_1 = K_2$, (3b)
by successively applying (3a), and (3c) by using $K_2 = D_n$. (3) for $T^n_k$ was also noticed independently by Koldobsky, but never published. For $BP^n_k$, (4) and (3b) for $k = 1$ were proved by Grinberg and Zhang in [16]. In the same paper, a very useful characterization of the class $BP^n_k$ was given (see Section 2). Combining it with (3) and (3c), we get as a corollary the following non-trivial result, which is of independent interest:

**Ellipsoid Corollary.** For any $1 \leq k \leq n - 1$ and $k$ ellipsoids $\{E_i\}_{i=1}^k$ in $\mathbb{R}^n$, define the body $L$ by:

$$\rho_L = \rho_{E_1} \cdot \ldots \cdot \rho_{E_k},$$

and let $k \leq l \leq n - 1$. Then there exists a sequence of star-bodies $\{L_i\}$ which tends to $L$ in the radial metric and satisfies:

$$\rho_{L_i} = \rho^j_{E_{i1}} + \ldots + \rho^j_{E_{ik}},$$

where $\{E_{ij}\}$ are ellipsoids.

Naturally, the case $E_1 = \ldots = E_k$ is of particular interest. In the same spirit, we give a strengthened version of Grinberg and Zhang’s characterization of $BP^n_k$ in Section 3. We remark that (3) from the Structure Theorem may in fact be a characterization of the classes $T^n_k$ or $BP^n_k$ for $k > 1$. In other words, it may be that for $C = BP$ or $C = I$, $L \in C^n_k$ iff there exist $\{K_i\}_{i=1}^k \subset C^n_1$, such that $\rho^k_L = \rho_{K_1} \cdot \ldots \cdot \rho_{K_k}$. Since in either case $C^n_1$ is the class of intersection bodies in $\mathbb{R}^n$, a proof of such a characterization for $C = I$ and a fixed $k$ would imply that $BP^n_k = T^n_k$ for that $k$.

In order to prove (3) for $C = BP$, we derive (what seems to be) a new formula for integration on products of Grassmann manifolds. The complete formulation and proof are given in the Appendix. A very similar formulation of the case $k_1, \ldots, k_r = 1$ was given by Blaschke and Petkantschin (in [30],[26] for an easy derivation), and used by Grinberg and Zhang in [16] to deduce that $BP^n_k \subset BP^n_1$ for all $1 \leq l \leq n - 1$. For $F \in G(n, n - l)$ and $1 \leq k < l \leq n - 1$, we denote by $G_{F}(n, n - k)$ the manifold $\{E \in G(n, n - k) | F \subset E\}$. The volume of the parallelepiped mentioned in the statement below is defined in the Appendix. A simplified formulation then reads as follows:

**Integration on products of Grassmann manifolds.** Let $n > 1$ be fixed. For $i = 1, \ldots, r$, let $k_i \geq 1$ denote integers whose sum $l$ satisfies $l \leq n - 1$. For $a = 1, \ldots, n$ denote by $G^a = G(n, n - a)$, and by $\mu^a$ the Haar probability measure on $G^a$. For $F \in G^l$ and $a = 1, \ldots, l - 1$, denote by $\mu^a_F$ the Haar probability measure on $G^a_F$. Denote by $E = (E_1, \ldots, E_r)$ an ordered set with $E_i \in G^{k_i}$. Then for any continuous function $f(E) = f(E_1, \ldots, E_r)$ on $G^{k_1} \times \ldots \times G^{k_r}$:

$$\int_{E_1 \in G^{k_1}} \cdots \int_{E_r \in G^{k_r}} f(E) d\mu^{k_1}(E_1) \cdots d\mu^{k_r}(E_r) =$$

$$\int_{F \in G^l} \int_{E_1 \in G^{k_1}_F} \cdots \int_{E_r \in G^{k_r}_F} f(E) \Delta(E) d\mu^{k_1}_F(E_1) \cdots d\mu^{k_r}_F(E_r) d\mu^l(F),$$

where $\Delta(E) = C_{n,\{k_i\}_1} \Omega(E)^{n-l}$, $C _{n,\{k_i\}_1}$ is a constant depending only on $n, \{k_i\}_1$, and $\Omega(E)$ denotes the $l$-dimensional volume of the parallelepiped spanned by unit volume elements of $E_1^\perp, \ldots, E_r^\perp$. 

1. GENERALIZED INTERSECTION BODIES
In Section 4 we attempt to bridge the gap between the the languages of Integral Geometry and Fourier Transforms, by establishing several new identities. As a by-product, we show, for instance, that \( \text{Ker} R_{n-k}^n = \text{Ker}(I \circ R_k)^n \), where \( I : C(G(n, k)) \to C(G(n, n-k)) \) denotes the operator defined as \( I(f)(E) = f(E^\perp) \). Essentially using the latter result, we show the following equivalence:

**Equivalence between \( k \) and \( n-k \).**

\[ \mathcal{B}P_k^n = I_k^n \iff \mathcal{B}P_{n-k}^n = I_{n-k}^n. \]

In Section 5 we try to attack the \( \mathcal{B}P_k^n = I_k^n \) question using the results of the previous sections together with a functional analytic approach. Our results indicate that this question is deeply connected to several fundamental questions in Integral Geometry concerning the structure of the Grassmann manifold. Let \( C_+(S^{n-1}) \) denote the set of non-negative continuous functions on the sphere, and let \( R_{n-k}(C(S^{n-1}))_+ \) denote the set of non-negative functions in the image of \( R_{n-k} \). Let \( \mathbb{A} \) denote the closure of a set \( A \) in the corresponding normed space. If \( \mu \in \mathcal{M}(G(n, n-k)) \), let \( \mu^\perp \in \mathcal{M}(G(n, k)) \) denote the measure defined by \( \mu^\perp(A) = \mu(A^\perp) \) for any Borel set \( A \subset G(n, k) \), where \( A^\perp = \{ E^\perp | E \in A \} \).

Fixing \( n \) and \( 1 \leq k \leq n-1 \), the main result of Section 5 is the following:

**Equivalence Theorem.** The following statements are equivalent:

1. **Equivalence of generalizations of intersection-bodies.**
   \[ \mathcal{B}P_k^n = I_k^n. \]

2. **Characterization of non-negative range of \( R_{n-k} \).**
   \[ R_{n-k}(C(S^{n-1}))_+ = R_{n-k}(C_+(S^{n-1})) + I \circ R_k(C_+(S^{n-1})). \] (1.4)

3. **A Negation Statement.**
   There does not exist a non-negative measure \( \mu \in \mathcal{M}(G(n, n-k)) \) such that \( R_{n-k}^n(d\mu) \geq 1 \) and \( R_k^k(d\mu^\perp) \geq 1 \) (where “\( \nu \geq 1 \)” means that \( \nu - 1 \) is a non-negative measure), and such that:
   \[ \inf \{ \langle \mu, f \rangle | f \in R_{n-k}(C(S^{n-1}))_+ \text{ and } \langle 1, f \rangle = 1 \} = 0. \]

The approach developed in Section 4 easily shows (once again) that \( \mathcal{B}P_k^n \subset I_k^n \). Analogously, it will be evident that the right hand side of (1.4) is a subset of the left hand side.

We will say that a set \( Z \subset G(n, n-k) \) satisfies the covering property if:

\[ \bigcup_{E \in Z} E \cap S^{n-1} = S^{n-1} \text{ and } \bigcup_{E \in Z} E^\perp \cap S^{n-1} = S^{n-1}. \] (1.5)

The following natural conjecture is given in Section 5 (see Lemma 5.7 and Remark 5.4):

**Covering Property Conjecture.** For any \( n > 0 \), \( 1 \leq k \leq n-1 \), if \( Z \subset G(n, n-k) \) is a closed set satisfying \( \bigcup_{E \in Z} E \cap S^{n-1} = S^{n-1} \), then there exists a non-negative measure \( \mu \in \mathcal{M}(G(n, n-k)) \) supported in \( Z \), such that \( R_{n-k}^n(d\mu) \geq 1 \).

Using this conjecture, we extend formulations (1)-(3) from the Equivalence Theorem in the following:

**Weak Equivalence Theorem.** The following statements are equivalent to each other:
see from general principles of Functional Analysis that (2) is the most elegant formulation, and perhaps the most natural for Integral Geometers. Without a doubt, Assuming the Covering Property Conjecture, formulations (1)-(3) imply (4)-(5).

Acknowledgments. I would like to deeply thank my supervisor Prof. Gideon Schechtman for helpful information and references about Radon Transforms. I also thank Alexander Koldobsky for going over the manuscript and for his helpful remarks. I also thank Prof. Semyon Alesker for many informative discussions, carefully reading the manuscript, and especially for believing in me and allowing me to pursue my interests. I would also like to thank Prof. Semyon Alesker for helpful information and references about Radon Transforms.

1. GENERALIZED INTERSECTION BODIES

(4) “Injectivity” of the Restricted Radon Transform.

For any $g \in R_{n-k}(C(S^{n-1}))_+$, if $Z = g^{-1}(0)$ satisfies the covering property then $g = 0$.

(5) Existence of barely balanced measures.

For any closed $Z \subset G(n,n-k)$ with the covering property, there exists a measure $\mu \in \mathcal{M}(G(n,n-k))$ such that $\mu|_{Z^c} \geq 1$ and $R_{n-k}^*(d\mu) = 0$.

Assuming the Covering Property Conjecture, formulations (1)-(3) imply (4)-(5).

For us, the formulation in (5) seems to have the most potential for understanding this problem, although we have not been able to advance in this direction. Without a doubt, (2) is the most elegant formulation, and perhaps the most natural for Integral Geometers.

We conclude by proposing another natural problem in Integral Geometry. Consider the operator $V_k : C(G(n,k)) \to C(G(n,k))$ defined as $V_k = I \circ R_{n-k} \circ R_k^*$. It is easy to see from general principles of Functional Analysis that $Ker V_k$ is orthogonal to $\overline{tmV_k}$, and therefore as an operator from $\overline{tmV_k}$ to itself, $V_k$ is injective and onto a dense set. We show in Section 4 that in addition, $V_k$ is self-adjoint. In the case $k = 1$, $C(G(n,1))$ may be identified with the class of even continuous functions on the sphere $C_e(S^{n-1})$, in which case $V_1 : C_e(S^{n-1}) \to C_e(S^{n-1})$ becomes the classical Spherical Radon Transform $R$ given by (1.1). Elegant inversion formulas for $V_1$ have been developed by many authors (see [17] and also [33], [14], [15], [31], [28]). Is it possible to do the same for the general $V_k$?

Acknowledgments. I would like to deeply thank my supervisor Prof. Gideon Schechtman for many informative discussions, carefully reading the manuscript, and especially for believing in me and allowing me to pursue my interests. I would also like to thank Prof. Alexander Koldobsky for going over the manuscript and for his helpful remarks. I also thank Prof. Semyon Alesker for helpful information and references about Radon Transforms.

2. Additional Notations and Previous Results

In this section we present some previously known results which will be useful for us later on. For completeness, we try to at least sketch the proofs of the main results, and on some occasions, provide alternative proofs. We also add several useful notations along the way.

2.1. Additional Notations. Let $G$ denote any locally compact topological space. The spaces of continuous and non-negative continuous real-valued functions on $G$ will be denoted by $C(G)$ and $C_+(G)$, respectively. When $G$ has a natural involution operator “$\sim$”, we will denote by $C_e(G)$ the space of continuous even functions on $G$. Whenever it makes sense, we will denote by $C^\infty(G)$ the space of infinitely smooth real-valued functions on $G$, and define $C^\infty_+(G)$ and $C^\infty_+(G)$ accordingly. Similarly, the spaces of signed and non-negative finite Borel measures on $G$ will be denoted by $\mathcal{M}(G)$ and $\mathcal{M}_+(G)$, respectively. When a natural involution operator “$\sim$” exists, the spaces $\mathcal{M}_e(G)$ and $\mathcal{M}_+(G)$ will denote the corresponding spaces of even measures. A measure $\mu$ is called even if $\mu(A) = \mu(-A)$ for every Borel set $A \subset G$. For $\mu \in \mathcal{M}(G)$ and $f \in C(G)$, we denote by $\langle \mu, f \rangle$ the action of the measure $\mu$ on $f$ as a linear functional. Whenever it is clear from the context what the underlying space $G$ is, we will write $\langle \mu, f \rangle$ instead of $\langle \mu, f \rangle_G$. 
1. GENERALIZED INTERSECTION BODIES

We will always assume that a fixed Euclidean structure is given on \( \mathbb{R}^n \), and denote by \( |x| \) the Euclidean norm of \( x \in \mathbb{R}^n \). We will denote by \( O(n) \) the group of orthogonal rotations in \( \mathbb{R}^n \). The group of volume-preserving linear transformations in \( \mathbb{R}^n \) will denoted by \( SL(n) \). For \( T \in SL(n) \), we denote \( T^{-1} = (T^{-1})^* \).

We will always use \( \sigma \) to denote the Haar probability measure on \( S^{n-1} \). \( G(n,0) \) and \( G(n,n) \) will denote the trivial atomic manifolds, and these are equipped of course with the trivial Haar probability measure.

For a star-body \( K \) (not necessarily convex), we define its Minkowski functional as \( \|x\|_K = \min \{ t \geq 0 \mid x \in tK \} \). When \( K \) is a centrally-symmetric convex body, this of course coincides with the natural norm associated with it. Obviously \( \rho_K(\theta) = \|\theta\|_K^{-1} \) for \( \theta \in S^{n-1} \).

2.2. Closure under basic operations. It is not hard to check from the definitions that the classes \( \mathcal{BP}_k^n \) and \( \mathcal{I}_k^n \) are closed under \( k \)-radial sums, full-rank linear transformations and limit in the radial metric. Indeed, the closure under limit in the radial metric follows from the definitions of \( \mathcal{I}_k^n \) and from the \( w^* \)-compactness of the unit ball of \( \mathcal{M}(G(n,n-k)) \) for \( \mathcal{BP}_k^n \). The closure under \( k \)-radial sums is also immediate for \( \mathcal{BP}_k^n \), but for \( \mathcal{I}_k^n \) this requires a little more thought. Indeed, by polar integration, if \( K_i \) is a \( k \)-intersection body of a star-body \( L_i \), for \( i = 1, 2 \), then the body \( K \) which is the \( k \)-radial sum of \( K_1 \) and \( K_2 \) is a \( k \)-intersection body of the \( n-k \)-radial sum of \( L_1 \) and \( L_2 \), and the general case follows by passing to a limit. The closure under full-rank linear-transformations requires a little more ingenuity. It is not so hard to check that if \( K \) is a \( k \)-intersection body of a star-body \( L \) then \( T(K) \) is a \( k \)-intersection body of \( T^{-1}(L) \) for \( T \in SL(n) \), which settles the case of \( \mathcal{I}_k^n \).

For \( \mathcal{BP}_k^n \), this requires additional work, and is actually a good exercise to show directly. Instead, we prefer to trivially deduce this from Theorem 2.1 below.

2.3. The class \( \mathcal{BP}_k^n \). The following characterization of \( \mathcal{BP}_k^n \), first proved by Goodey and Weil in [12] for intersection-bodies (the case \( k = 1 \)), and extended to general \( k \) by Grinberg and Zhang in [16], is extremely useful:

**Theorem 2.1** (Grinberg and Zhang). A star-body \( K \) is a \( k \)-Busemann-Petty body iff it is the limit of \( \{ K_i \} \) in the radial metric, where each \( K_i \) is a finite \( k \)-radial sums of ellipsoids \( \{ E_j \} \):

\[
\rho_k^K = \rho_{E_{i_1}}^k + \ldots + \rho_{E_{i_m}}^k .
\]

Before commenting on the proof of this theorem, we introduce the following useful notion used by Grinberg and Zhang. For any \( G \), a homogeneous space of \( O(n) \), and measures \( \mu \in \mathcal{M}(G) \) and \( \eta \in \mathcal{M}(O(n)) \), we define their convolution \( \eta * \mu \in \mathcal{M}(G) \) as the measure satisfying \( \eta * \mu(A) = \int_{O(n)} \mu(u^{-1}A) \eta(du) \) for every Borel subset \( A \subset G \). The definition is essentially the same when \( \eta \in \mathcal{M}(H) \), where \( H \) is another homogeneous space of \( O(n) \), by identifying between \( \eta \) and its lifting \( \tilde{\eta} \in \mathcal{M}(O(n)) \) defined as \( \tilde{\eta}(A) = \eta(\pi(A)) \) for any Borel subset \( A \subset O(n) \), where \( \pi : O(n) \to H \) is the canonical projection.

Let \( \sigma_F \) denote the Haar probability measure on \( S^{n-1} \cap F \), so that as a linear functional, for any \( f \in C(S^{n-1}) \), \( \sigma_F(f) = R_{n-k}(f)(F) \). The key idea underlying Theorem 2.1 is an important observation: for any \( F \in G(n,n-k) \), one may explicitly construct a family of ellipsoids \( \{ E_i(F,\epsilon) \} \), such that \( \rho_{E_i(F,\epsilon)}^k \) tends to \( \sigma_F \) in the \( w^* \)-topology (as \( \epsilon \to 0 \)).
ellipsoid $E_i(F, \epsilon)$ is defined by:

$$\|x\|^2_{E_i(F, \epsilon)} = \frac{|\text{Proj}_F(x)|^2}{a(\epsilon)^2} + \frac{|\text{Proj}_{F^\perp}(x)|^2}{b(\epsilon)^2},$$

where $\text{Proj}_E$ denotes the orthogonal projection onto $E$, and $a(\epsilon), b(\epsilon)$ are chosen appropriately. As observed by Grinberg and Zhang, one may write $R_{n-k}^*(d\mu) = \mu * \sigma_{F_0}$, where $F_0 = \pi(\epsilon)$, $\pi$ is the identity element in $O(n)$ and $\sigma$ is the canonical projection as above. Since in the $w^*$-topology, $\sigma_{F_0}$ may be approximated by $\rho^k_{E_i(F_0, \epsilon)}$, and $\mu$ by a discrete measure, the Theorem follows after several technicalities are treated.

We mention a different way to conclude the theorem. It is easy to verify that:

$$R_{n-k}(\rho^k_{E_i(F, \epsilon)})\langle E \rangle = R_{n-k}(\rho^k_{E_i(F, \epsilon)})(F) \forall E, F \in G(n, n - k).$$

Denoting $G = G(n, n - k)$ for short, if $\rho^k_K = R_{n-k}^*(d\mu)$ then:

$$R_{n-k}(\rho^k_K)(F) = \int_{S^{n-1}} \rho^k_K(\theta)d\sigma_F(\theta) = \lim_{\epsilon \to 0} \int_{S^{n-1}} \rho^k_{E_i(F, \epsilon)}(\theta) R_{n-k}^*(d\mu)(\theta)d\sigma(\theta) = \lim_{\epsilon \to 0} \int_G R_{n-k}(\rho^k_{E_i(F, \epsilon)})(E)d\mu(E) = \lim_{\epsilon \to 0} \int_G R_{n-k}(\rho^k_{E_i(F, \epsilon)})(E)d\mu(E) = R_{n-k} \left( \lim_{\epsilon \to 0} \int_G \rho^k_{E_i(F, \epsilon)}(E)d\mu(E) \right)(F),$$

where we have used the uniform convergence of all the limits involved and that $R_{n-k}$ is a continuous operator w.r.t. the maximum-norm. The result then follows from the injectivity of $R_{n-k}$ on $C_0(S^{n-1})$.

Grinberg and Zhang’s characterization of the class $\mathcal{BP}_k^n$ implies that it is actually generated from $D_n$, the Euclidean unit Ball, by taking full-rank linear transformations, $k$-radial sums, and limit in the radial metric. By starting from any other star-body $L$ and performing these operations, it is obvious that $D_n$ may be constructed, and therefore we see that $\mathcal{BP}_k^n$ is the minimal non-empty class which is closed under these three operations. Since $T_k^n$ trivially contains $D_n$ and is also closed under these operations, it immediately follows that:

**Corollary 2.2.** $\mathcal{BP}_k^n \subset T_k^n$.

This was first observed by Koldobsky in [21] using a different approach. We will give another proof of this in Corollary 4.4, which is in a sense more concrete.

We conclude this preliminary discussion of the class $\mathcal{BP}_k^n$ by elaborating a little more on the operation of convolution between measures on homogeneous spaces of $O(n)$. Let $G, H$ denote homogeneous spaces of $O(n)$. We identify between a function $f \in C(G)$ and the measure on $C(G)$ whose density w.r.t. the Haar probability measure on $G$ is given by $f$, and consider expressions of the form $f * \mu$ and $\mu * f$ for $\mu \in \mathcal{M}(H)$. With the same notations, if $f \in C^\infty(G)$ then a standard argument shows that $f * \mu \in C^\infty(H)$ and that $\mu * f \in C^\infty(G)$. If $\eta \in \mathcal{M}(O(n))$, it is immediate to check that $\langle \mu, \eta * f \rangle_G = \langle \eta^{-1} * \mu, f \rangle_G$, where $\eta^{-1} \in \mathcal{M}(O(n))$ is the measure defined by $\eta^{-1}(A) = \eta(\pi^{-1}(A))$ and $\pi^{-1} = \{ u^{-1} | u \in A \}$ for a Borel set $A \subset O(n)$. If $\mu_i \in \mathcal{M}(G_i)$ for $i = 1, 2, 3$, one may verify that this operation is associative: $(\mu_1 * \mu_2) * \mu_3 = \mu_1 * (\mu_2 * \mu_3)$. We conclude with the following lemma from [16] which will be useful later on.
Lemma 2.3. There exists a sequence of functions \( \{u_i\} \subset C^\infty_+ (O(n)) \) called an approximate identity, such that for any homogeneous space \( G \) of \( O(n) \):

1. For any \( \mu \in \mathcal{M}(G) \), \( u_i * \mu \in C^\infty(G) \) tends to \( \mu \) in the \( \mathcal{w}^* \)-topology.
2. For any \( g \in C(G) \), \( u_i * g \in C^\infty(G) \) tends to \( g \) uniformly.

2.4. The class \( T_0^p \). In order to handle the class \( T_0^p \), we shall need to adopt a technique extensively used by Koldobsky: Fourier transforms of homogeneous distributions. We will only outline the main ideas here, usually omitting the technical details - we refer the reader to [22] for those. We denote by \( S'(\mathbb{R}^n) \) the space of rapidly decreasing infinitely differentiable test functions in \( \mathbb{R}^n \), and by \( S' \) the space of distributions over \( S(\mathbb{R}^n) \). The Fourier Transform \( \hat{f} \) of a distribution \( f \in S' \) is defined by \( (\hat{f}, \phi) = (f, \hat{\phi}) \) for every test function \( \phi \), where \( \hat{\phi}(y) = \int \phi(x) \exp(-iy(x,y))dx \). A distribution \( f \) is called homogeneous of degree \( p \in \mathbb{R} \) if \( (f, \phi(t \cdot)) = |t|^{n+p} (f, \phi) \) for every \( t > 0 \), and it is called even if the same is true for \( t = -1 \). An even distribution \( f \) always satisfies \( (\hat{f})^\wedge = (2\pi)^n f \). The Fourier Transform of an even homogeneous distribution of degree \( p \) is an even homogeneous distribution of degree \( -n - p \). A distribution \( f \) is called positive if \( (f, \phi) \geq 0 \) for every \( \phi \geq 0 \), implying that \( f \) is necessarily a non-negative Borel measure on \( \mathbb{R}^n \). We use Schwartz’s generalization of Bochner’s Theorem ([10]) as a definition, and call a homogeneous distribution positive-definite if its Fourier transform is a positive distribution.

Before proceeding, let us give some intuition about how the Fourier transform of a homogeneous continuous function looks like. Because of the homogeneity, it is enough to consider a continuous function on the sphere \( f \in C(S^{n-1}) \), and take its homogeneous extension of degree \( p \in \mathbb{R} \), denoted \( E_p(f) \), to the entire \( \mathbb{R}^n \) (formally excluding \( \{0\} \) if \( p < 0 \). When \( p > -n \), the function \( E_p(f) \) is locally integrable, and its action as a distribution on a test function \( \phi \) is simply by integration. Passing to polar coordinates, we have:

\[
(\hat{E_p(f)}, \phi) = \int_{S^{n-1}} f(\theta) \int_0^\infty r^{n+p-1} \phi(r\theta) dr d\theta.
\]

When \( p \leq -n \), we can no longer interpret the action of \( E_p(f) \) as an integral. Fortunately, we will mainly be concerned with Fourier transforms of continuous functions which are homogeneous of degree \( p \in (-n, 0) \). This ensures that the Fourier transform is a homogeneous distribution of degree \( -p - n \), which is in the same range \( (-n, 0) \). Note that the resulting distribution need not necessarily be a continuous function on \( \mathbb{R}^n \setminus \{0\} \), nor even a measure on \( \mathbb{R}^n \) (although this will not occur in our context). We will denote by \( E_p^\wedge(f) \) the Fourier transform of \( E_p(f) \). In order to ensure that \( E_p^\wedge(f) \) is a continuous function, we need to add some smoothness assumptions on \( f \) ([22]). We remark that for a continuous function \( f \in C(S^{n-1}) \), \( E_p^\wedge(f) \) is always continuous for \( p \in (-n, n+1] \), and that for an infinitely smooth \( f \in C^\infty(S^{n-1}) \), \( E_p^\wedge(f) \) is infinitely smooth for any \( p \in (-n, 0) \). Whenever \( E_p^\wedge(f) \) is continuous on \( \mathbb{R}^n \setminus \{0\} \), it is uniquely determined by its value on \( S^{n-1} \) (by homogeneity). In that case, by abuse of notation, we identify between \( E_p^\wedge(f) \) and its restriction to \( S^{n-1} \), and in particular, consider \( E_p^\wedge \) as an operator from \( C^\infty(S^{n-1}) \) to \( C^\infty(S^{n-1}) \).

When \( f = 1 \), it is easy to verify that \( E_p^\wedge(1) \) is rotational invariant, so by the homogeneity, it must be a multiple of \( E_{-n-p}(1) \). For a rigorous proof we refer to [10, p. 192], and state this for future reference as:
Lemma 2.4. Fix \( n \) and let \( p \in (0, n) \). Then:
\[
E_{-p}^\wedge(1) = c(n,p)E_{-n+p}(1) \quad \text{where} \quad c(n,p) = \pi^{n/2}2^{n-p} \frac{\Gamma\left(\frac{n-p}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}.
\]
Since \((E_{-p}^\wedge(1))^\wedge = (2\pi)^n E_{-p}(1)\), it is clear that:
\[
c(n,p)c(n,n-p) = (2\pi)^n.
\]

The following characterization was given by Koldobsky in [21]:

Theorem 2.5 (Koldobsky). The following are equivalent for a centrally-symmetric starbody \( K \) in \( \mathbb{R}^n \):

1. \( K \) is a \( k \)-intersection body.
2. \(|x|_K^{-k} \) is a positive definite distribution on \( \mathbb{R}^n \), meaning that its Fourier-transform \((|||_K^{-k})^\wedge \) is a non-negative Borel measure on \( \mathbb{R}^n \).
3. The space \((\mathbb{R}^n, \|\cdot\|_K)\) embeds in \( L_{-k} \).

For completeness, we briefly give the definition of embedding in \( L_{-k} \), although we will not use this later on. Let us denote the class of centrally-symmetric star bodies \( K \) in \( \mathbb{R}^n \) for which \((\mathbb{R}^n, \|\cdot\|_K)\) embeds in \( L_p \) by \( SL^n_p \). For \( p > 0 \), it is well known (e.g. [21]) that \( K \in SL^n_p \) iff:

\[
\|x\|_K^p = \int_{S^{n-1}} |\langle x, \theta \rangle|^p \, d\mu_K(\theta),
\]
for some \( \mu_K \in \mathcal{M}_+(S^{n-1}) \). Unfortunately, this characterization breaks down at \( p = -1 \) since the above integral no longer converges. However, Koldobsky showed that it is possible to regularize this integral by using Fourier-transforms of distributions, and gave the following definition: \((\mathbb{R}^n, \|\cdot\|_K)\) embeds in \( L_{-p} \) for \( 0 < p < n \) iff there exists a measure \( \mu_K \in \mathcal{M}_+(S^{n-1}) \) such that for any even test-function \( \phi \):

\[
\int_{\mathbb{R}^n} \|x\|_K^{-p} \phi(x) \, dx = \int_{S^{n-1}} \int_0^\infty t^{p-1} \phi(t\theta) \, dt \, d\mu_K(\theta).
\]

Let us review the statements of Theorem 2.5. (2) is an extremely useful characterization of \( k \)-intersection bodies, and immediately implies the closure of \( I^n_k \) under the standard three operations. Characterization (3) provides additional motivation for why it is reasonable to believe that \( \mathcal{B}P^n_k = I^n_k \). For \( p \neq 0 \), the \( p \)-norm sum of two bodies \( L_1, L_2 \) is defined as the body \( L \) satisfying \( \|\cdot\|_L^p = \|\cdot\|_{L_1}^p + \|\cdot\|_{L_2}^p \). We will denote by \( D^n_p \), the class of bodies created from \( D_n \) by applying full-rank linear-transformations, \( p \)-norm sums, and taking the limit in the radial metric. Using the characterization in (2.1), it is easy to show (e.g. [16, Theorem 6.13]) that for \( p > 0 \), the class \( SL^n_p \) coincides with \( D^n_p \). Although this characterization breaks down at \( p = -1 \), it is still reasonable to expect that the property \( SL^n_p = D^n_p \) should pass over to negative values of \( p \) when \( SL^n_p \) is (in some sense) extended to this range and becomes \( SL^n_{-k} = I^n_k \). But by Grinberg and Zhang’s characterization (Theorem 2.1), this is exactly satisfied by \( k \)-Busemann-Petty bodies: \( \mathcal{B}P^n_k = D^n_{-k} \). This suggests that indeed \( \mathcal{B}P^n_k = I^n_k \).

In addition to the characterization (3) of \( I^n_k \) as the class of unit-balls of subspaces of \textit{scalar} \( L_{-k} \) spaces, a functional analytic characterization of \( \mathcal{B}P^n_k \) as the class of unit-balls of
subspaces of vector valued $L_{-k}$ spaces (in a manner similar to (2.2)), was given in [21]. This provides additional motivation for believing that $\mathcal{B}P^n_k = T^n_k$, as this would be an extension to negative values of $p$ of the fact that every separable vector valued $L_p$ space is isometric to a subspace of a scalar $L_p$ space and vice-versa.

We proceed to explain why (1) and (2) in Theorem 2.5 are equivalent. To this end, we will need the following Spherical Parseval identity, due to Koldobsky ([22]):

Spherical Parseval (Koldobsky). Let $f, g \in C^\infty_e(S^{n-1})$, and $p \in (0, n)$. Then:

$$
\int_{S^{n-1}} E_{-p}^\wedge(f)(\theta)E_{-n+p}^\wedge(g)(\theta)d\sigma(\theta) = (2\pi)^n \int_{S^{n-1}} f(\theta)g(\theta)d\sigma(\theta).
$$

We prefer to present a self-contained proof of this identity, which seems simpler than the previous approaches in [22].

**Proof.** Let $f = \sum_{k=0}^{\infty} f_k$ and $g = \sum_{k=0}^{\infty} g_k$ be the canonical decompositions into spherical harmonics, where $f_k, g_k \in H_k$ and $H_k$ is the space of spherical harmonics of degree $k$. Since $f$ and $g$ are even, it follows that $f_{2k+1} = g_{2k+1} = 0$. It is well known ([32]) that for $q \in (-n, 0)$, the linear operator $E_q^\wedge : C^\infty(S^{n-1}) \to C^\infty(S^{n-1})$ decomposes into a direct sum of scalar operators acting on $H_k$. Indeed, one only needs to check that the $H_k$’s are eigenspaces of $E_q^\wedge$, and by Schur’s Representation Lemma and the fact that the Fourier transform commutes with the action of the orthogonal group, it follows that $E_q^\wedge$ must act as a scalar on these spaces. Denote by $c_k^{(q)}$ the eigenvalue satisfying $E_q^\wedge(h_k) = c_k^{(q)} h_k$ for any $h_k \in H_k$. The exact value of $c_k^{(q)}$ is well known ([32, Theorem 4.1]), but is irrelevant to our proof. It remains to notice that since:

$$E_{-n+p}^\wedge(E_{-p}^\wedge(f)) = (E_{-p}^\wedge(f))^\wedge |_{S^{n-1}} = (2\pi)^n f,$$

for any $f \in C^\infty_e(S^{n-1})$, we must have $c_k^{(-n+p)} c_k^{(-p)} = (2\pi)^n$ for all even $k$’s. Using the fact that spherical harmonics of different degrees are orthogonal to each other in $L_2(S^{n-1})$, and that $f, g, E_{-p}^\wedge(f)$ and $E_{-n+p}^\wedge(g)$ are all in $L_2(S^{n-1})$, we conclude:

$$
\int_{S^{n-1}} E_{-p}^\wedge(f)(\theta)E_{-n+p}^\wedge(g)(\theta)d\sigma(\theta) = \int_{S^{n-1}} \sum_{k=0}^{\infty} c_k^{(-p)} f_k(\theta) \sum_{l=0}^{\infty} c_l^{(-n+p)} g_l(\theta)d\sigma(\theta)
$$

$$= \int_{S^{n-1}} \sum_{k=0}^{\infty} c_k^{(-p)} \sum_{l=0}^{\infty} f_k(\theta)g_l(\theta)d\sigma(\theta) = (2\pi)^n \int_{S^{n-1}} \sum_{k=0}^{\infty} f_k(\theta)g_k(\theta)d\sigma(\theta)
$$

$$= (2\pi)^n \int_{S^{n-1}} f(\theta)g(\theta)d\sigma(\theta).$$

\[ \square \]

Note that the above argument actually shows that the Spherical Parseval identity is also valid when $f, g, E_{-p}^\wedge(f), E_{-n+p}^\wedge(g) \in L_2(S^{n-1})$. 
Remark 2.1. Applying the theorem to \( g = E^\wedge_{n+p}(g') \) for \( g' \in C^\infty_c(S^{n-1}) \) and using that \( E^\wedge_{-n+p}(g) = (2\pi)^n g' \), we note that the Spherical Parseval identity has the following equivalent form, which we will sometimes use:

\[
\int_{S^{n-1}} E^\wedge_{-p}(f)(\theta) g(\theta) d\sigma(\theta) = \int_{S^{n-1}} f(\theta) E^\wedge_{-p}(g)(\theta) d\sigma(\theta).
\]

Another useful result due to Koldobsky, which looks very similar to the Spherical Parseval identity, is the following:

**Theorem 2.6** (Koldobsky). Let \( f \in C^\infty_c(S^{n-1}) \), and let \( k = 1, \ldots, n-1 \). Then for any \( H \in G(n, k) \):

\[
\int_{S^{n-1} \cap H^\perp} E^\wedge_{-k}(f)(\theta) d\sigma_{H^\perp}(\theta) = c(n, k) \int_{S^{n-1} \cap H} f(\theta) d\sigma_H(\theta),
\]

where \( c(n, k) \) is the constant from Lemma 2.4.

Informally, the latter Theorem may be considered as a special case of the Spherical Parseval identity, by setting \( g = d\sigma_H \) and verifying that in the appropriate sense \( E^\wedge_{-n+k}(d\sigma_H) = c(n, k) d\sigma_{H^\perp} \). The constant in front of the right hand integral is verified by choosing \( f = 1 \) and using Lemma 2.4. One way to make this argument work is to use Grinberg and Zhang’s approximation of \( d\sigma_H \) by the functions \( \rho_k^{n-k} \), which when written as \( ||\cdot||^{-n+k} \) are seen to be already homogeneous of degree \( -n + k \). Computing the Fourier transform is particularly easy, since \( \mathcal{E}_i = T_i(D_n) \), and therefore:

\[
(||\cdot||_{T_i(D_n)}^{-n+k})^\wedge(x) = (||T_i^{-1}(\cdot)||_{D_n}^{-n+k})^\wedge(x) = det(T_i)(||\cdot||_{D_n}^{-n+k})^\wedge(T_i^*(x)) = det(T_i)d(n, k) ||T_i^*(x)||^{-k}_{T_i^{-1}(D_n)}.
\]

Using Grinberg and Zhang’s approximation again, it turns out that \( det(T_i)d(n, k)\rho_k^{n-k} \) tends in the \( w^* \)-topology to \( c(n, k) d\sigma_{H^\perp} \).

We can now sketch a proof of Koldobsky’s Fourier transform characterization of \( k \)-intersection bodies. By abuse of notation, when \( (||\cdot||_{K}^{-k})^\wedge \) is continuous, we will often use \( ||\cdot||_{K}^{-k} \) to indicate both locally integrable functions on \( \mathbb{R}^n \) and continuous functions on \( S^{n-1} \). By definition, an *infinitely smooth* star-body \( K \) which is a \( k \)-intersection body of a star-body \( L \), satisfies Vol \((K \cap H^\perp) = Vol \((L \cap H) \) for all \( H \in G(n, n - k) \). Passing to polar coordinates, this is equivalent to:

\[
R_k(||\cdot||_{K}^{-k})(H^\perp) = \frac{Vol(D_{n-k})}{Vol(D_k)} R_{n-k}(||\cdot||_{L}^{-n+k})(H) \quad \forall H \in G(n, n - k).
\]

But using Theorem 2.6, we see that:

\[
R_k(||\cdot||_{K}^{-k})(H^\perp) = c(n, k)^{-1} R_{n-k}(||\cdot||_{K}^{-k})^\wedge(H) \quad \forall H \in G(n, n - k).
\]

From the injectivity of \( R_{n-k} \) on \( C_c(S^{n-1}) \), it follows that:

\[
||\cdot||_{K}^{-k} = c(n, k)^{-1} \frac{Vol(D_{n-k})}{Vol(D_k)} ||\cdot||_{L}^{-n+k}
\]

on \( S^{n-1} \), and hence on all \( \mathbb{R}^n \) by homogeneity. We conclude that \( ||\cdot||_{K}^{-k} \) is a non-negative continuous function on \( \mathbb{R}^n \setminus \{0\} \), and hence positive as a distribution. For an arbitrary
star-body $K$ which is a $k$-intersection body of a star-body $L$, the same conclusion holds by approximation ($\left|\left\|\cdot\right\|_{K}^{-k}\right\|^\wedge$ is still continuous by the continuity of $\left|\left\|\cdot\right\|_{L}^{-n+k}\right|$). One may also invert the argument, proving that for a star-body $K$, if $\left|\left\|\cdot\right\|_{K}^{-k}\right\|^\wedge$ is a continuous function which is non-negative, then $K$ is a $k$-intersection body of a star-body $L$ (defined as above). Taking the limit in the radial metric, $\left|\left\|\cdot\right\|_{K}^{-k}\right\|^\wedge$ need not necessarily be a continuous function for a general $k$-intersection body $K$ which is the limit of the bodies $\{K_i\}$ (which are $k$-intersection bodies of star-bodies). Nevertheless, the non-negative continuous functions $\left|\left\|\cdot\right\|_{K_i}^{-k}\right\|^\wedge$ must satisfy:

$$\int_{S^{n-1}} \left|\left|\left\|\cdot\right\|_{K_i}^{-k}\right\|^\wedge(\theta)\right| d\sigma(\theta) = c(n, k) \int_{S^{n-1}} \left|\left|\theta\right\|_{K_i}^{-k}\right| d\sigma(\theta),$$

by the Spherical Parseval identity with $g = 1$ and Lemma 2.4, and therefore the integral on the left hand side is bounded. Using the compactness of the unit-ball of $\mathcal{M}(S^{n-1})$ in the $w^*$-topology, there must be an accumulation point of $\{\left|\left\|\cdot\right\|_{K}^{-k}\right\|^\wedge\}$, which is a non-negative Borel measure on $S^{n-1}$. This argument is the main idea in the proof that for a star-body $K$, $K \in \mathcal{I}_k^n$ iff $\left|\left\|\cdot\right\|_{K}^{-k}\right\|^\wedge$ is a non-negative Borel measure on $\mathbb{R}^n$.

When $K$ is infinitely smooth, we summarize this in the following alternative definition for $\mathcal{I}_k^n$, and use it instead of the original one:

**Alternative Definition of $\mathcal{I}_k^n$.** For an infinitely smooth star-body $K$, $K \in \mathcal{I}_k^n$ iff $\left|\left\|\cdot\right\|_{K}^{-k}\right\|^\wedge \geq 0$ as a $C^\infty$ function on $S^{n-1}$.

For a general star-body $K$, we will use Koldobsky’s characterization in the following spherical version, which is an immediate consequence of the above reasoning (a rigorous proof is given in [22, Corollary 3.23]):

**Proposition 2.7.** For a star-body $K$, $K \in \mathcal{I}_k^n$ iff there exists a non-negative Borel measure $\mu$ on $S^{n-1}$, such that for any $f \in C^\infty_c(S^{n-1})$:

$$\int_{S^{n-1}} f(\theta) \rho_K^n(\theta) d\sigma(\theta) = \int_{S^{n-1}} E_{n+k}^\wedge(f)(\theta) d\mu(\theta).$$

### 3. The Identical Structures of $\mathcal{BP}_k^n$ and $\mathcal{I}_k^n$

In this section we will prove the Structure Theorem, which was formulated in the Introduction. We will skip over item 1 which basically follows from the definitions, and was already explained in detail in Section 2. Item 2 also follows immediately: by definition, $\mathcal{I}_1^n = \mathcal{BP}_1^n$ is exactly the class of intersection bodies in $\mathbb{R}^n$; any star-body $K$ in $\mathbb{R}^n$ is an $n - 1$-intersection body of a star-body $L$, defined by $\rho_L(\theta) = 1/2\text{Vol}(K \cap \theta^\perp)$; and by definition, $R_1^n$ acts as the identity on $C_c(S^{n-1})$, hence $\nu_{K}^{n-1} = R_{K}^n(\nu_{K}^{n-1})$ for any star-body $K$, implying that $K \in \mathcal{BP}_n^{n-1}$. We therefore commence the proof from item 3. We will prove the Theorem for $\mathcal{BP}_k^n$ and $\mathcal{I}_k^n$ separately, because of the different techniques involved in the proof.

Before we start, we will need the following useful lemma, which appears implicitly in [16]. We denote by $\mathcal{BP}_k^{n,\infty}$ the class of star-bodies $K$ such that $\rho_K^k = R_{n-k}^n(g)$, where $g \in C^\infty_c(G(n, n-k))$. Obviously $\mathcal{BP}_k^{n,\infty} \subset \mathcal{BP}_k^n$. 

1. GENERALIZED INTERSECTION BODIES 27
Lemma 3.1 ([16]). $\mathcal{B}\mathcal{P}_{k}^{n, \infty}$ is dense in $\mathcal{B}\mathcal{P}_{k}^{n}$. In particular, the class of infinitely smooth bodies in $\mathcal{B}\mathcal{P}_{k}^{n}$ is dense in $\mathcal{B}\mathcal{P}_{k}^{n}$.

Proof. Let $K \in \mathcal{B}\mathcal{P}_{k}^{n}$, and assume that $\rho_{K}^{k} = R_{n-k}^{*}(d\mu)$ where $d\mu \in \mathcal{M}_{+}(G(n, n - k))$. Let $\{u_{i}\} \subset C^{\infty}(O(n))$ be an approximate identity as in Lemma 2.3. Let $K_{i}$ be the star-body for which $\rho_{K_{i}}^{k} = u_{i} \ast \rho_{K}^{k}$. Then by Lemma 2.3, $\{K_{i}\}$ is a sequence of infinitely smooth star-bodies which tend to $K$ in the radial metric. As in the proof of Theorem 2.1, we write $\rho_{K}^{k} = \mu \ast \sigma_{H_{0}}$, and therefore:

$$\rho_{K_{i}}^{k} = u_{i} \ast (\mu \ast \sigma_{H_{0}}) = (u_{i} \ast \mu) \ast \sigma_{H_{0}} = R_{n-k}^{*}(u_{i} \ast \mu).$$

Since $u_{i} \ast \mu \in C_{+}^{\infty}(G(n, n - k))$, this concludes the proof of the lemma. \qed

Remark 3.1. By the Lemma and the closure of $\mathcal{B}\mathcal{P}_{k}^{n}$ (for any $k = 1, \ldots, n - 1$) under limit in the radial metric, it is enough to prove all the remaining items for the classes $\mathcal{B}\mathcal{P}_{k}^{n, \infty}$.

We will also require the following notations. Given $F \in G(n, m)$ and $k \geq m$, we denote by $G_{F}(n, k)$ the manifold $\{E \in G(n, k) | F \subset E\}$. For $\theta \in S^{n-1}$ we identify between $\theta$ and the one-dimensional subspace spanned by it. $G_{F}(n, k)$ is a homogeneous space of $O(n)$, therefore there exists a unique Haar probability measure on $G_{F}(n, k)$, which is invariant to orthogonal rotations in $O(n)$ which preserve $F$. Thus, if we denote by $\nu_{\sigma}$ the Haar probability measure on $G_{\sigma}(n, m)$ for $\sigma \in S^{m-1}$, then for any $g \in C(G(n, m))$ we may write:

$$R_{m}^{*}(g)(\theta) = \int_{G_{\sigma}(n, m)} g(E)d\nu_{\sigma}(E).$$

We will need the following fact, which is an immediate corollary of Proposition 6.1. We postpone the formulation and proof of Proposition 6.1 for the Appendix, as the technique involved is different in spirit to the rest of this note.

Corollary 3.2. Let $n > 1$ and let $k_{1}, k_{2} \geq 1$ denote integers such that $l = k_{1} + k_{2} \leq n - 1$. Let $\theta \in S^{n-1}$. For $a = k_{1}, k_{2}, l$, denote by $G^{a} = G(n, n - a)$ and by $\mu_{\theta}^{a}$ the Haar probability measure on $G^{a}_{\theta}$. For $F \in G^{l}$ and $a = k_{1}, k_{2}$, denote by $\mu_{\theta}^{k_{1}, k_{2}}$ the Haar probability measure on $G^{k_{1}, k_{2}}_{\theta}$. Then for any continuous function $f(E_{1}, E_{2})$ on $G^{k_{1}} \times G^{k_{2}}$:

$$\int_{E_{1} \in G_{\theta}^{k_{1}}} \int_{E_{2} \in G_{\theta}^{k_{2}}} f(E_{1}, E_{2})d\mu_{\theta}^{k_{1}}(E_{1})d\mu_{\theta}^{k_{2}}(E_{2}) =$$

$$\int_{F \in G_{\theta}^{l}} \int_{E_{1} \in G_{\theta}^{k_{1}}} \int_{E_{2} \in G_{\theta}^{k_{2}}} f(E_{1}, E_{2})\Delta(E_{1}, E_{2})d\mu_{\theta}^{k_{1}}(E_{1})d\mu_{\theta}^{k_{2}}(E_{2})d\mu_{\theta}^{l}(F),$$

where $\Delta(E_{1}, E_{2})$ is some (known) non-negative continuous function on $G^{k_{1}} \times G^{k_{2}}$.

We will show the following basic property of $k$-Busemann-Petty bodies, and immediately deduce (3a), (3b) and (3c) from the Structure Theorem in the Introduction.

Proposition 3.3. Let $K_{1} \in \mathcal{B}\mathcal{P}_{k_{1}}^{n}$ and $K_{2} \in \mathcal{B}\mathcal{P}_{k_{2}}^{n}$ for $k_{1}, k_{2} \geq 1$ such that $l = k_{1} + k_{2} \leq n - 1$. Then the star-body $L$ defined by $\rho_{L}^{k_{1}, k_{2}} = \rho_{K_{1}, K_{2}}^{k_{1}} \ast \rho_{K_{2}}^{k_{2}}$, satisfies $L \in \mathcal{B}\mathcal{P}_{k_{1}}^{n}$.
Proof. First, assume that $K_i \in \mathcal{BP}_{k_i}^{n,\infty}$ for $i = 1, 2$, so that $\rho_{K_i}^k = R_{n-k_i}(g_i)$ with $g_i \in C_+^\infty(G(n, n - k_i))$. Using the notations and result of Corollary 3.2, we have:

$$\rho_{L}^k(\theta) = \rho_{K_1}^k(\theta)\rho_{K_2}^k(\theta) = \int_{E_1 \in G_\theta^{k_1}} g_1(E_1) d\mu_\theta^{E_1}(E_1) \int_{E_2 \in G_\theta^{k_2}} g_2(E_2) d\mu_\theta^{E_2}(E_2)$$

$$= \int_{F \in G_\theta} \int_{E_1 \in G_\theta^{k_1}} \int_{E_2 \in G_\theta^{k_2}} g(E_1) g(E_2) \Delta(E_1, E_2) d\mu_\theta^{E_1}(E_1) d\mu_\theta^{E_2}(E_2) d\mu_{\|F\|}(F).$$

Denoting:

$$h(F) = \int_{E_1 \in G_\theta^{k_1}} \int_{E_2 \in G_\theta^{k_2}} g(E_1) g(E_2) \Delta(E_1, E_2) d\mu_\theta^{E_1}(E_1) d\mu_\theta^{E_2}(E_2),$$

we see that $h(F)$ is a non-negative continuous function on $G(n, n - l)$. Therefore:

$$\rho_{L}^k(\theta) = \int_{F \in G_\theta} h(F) d\mu_\theta^{F}(F),$$

implying that $L \in \mathcal{BP}_{l}^{n}$. The general case, when $K_i \in \mathcal{BP}_{k_i}^{n}$ without any smoothness assumptions, follows from Remark 3.1. Indeed, by approximating each $K_i$ in the radial metric by smooth bodies $\{K_i^n\} \subset \mathcal{BP}_{k_i}^{n}$, the bodies $\{L^n\}$ defined by $\rho_{L}^k = \rho_{K_1}^k \rho_{K_2}^k$ satisfy that $L^n \in \mathcal{BP}_{l}^{n}$ and obviously $L^n$ approximate $L$ in the radial metric, implying that $L \in \mathcal{BP}_{l}^{n}$. 

Applying Proposition 3.3 with $K_1 = K_2$, we have:

**Corollary 3.4.** $\mathcal{BP}_{k_1}^n \cap \mathcal{BP}_{k_2}^n \subset \mathcal{BP}_{k_1+k_2}^n$ for $k_1, k_2 \geq 1$ such that $k_1 + k_2 \leq n - 1$.

By successively applying Corollary 3.4, we see that $\mathcal{BP}_{k}^n \subset \mathcal{BP}_{l}^{n}$ if $k$ divides $l$. The question whether $\mathcal{BP}_{k}^n \subset \mathcal{BP}_{l}^{n}$ for general $1 \leq k < l \leq n - 1$ remains open. Nevertheless, we are able to show the following "non-linear" embedding of $\mathcal{BP}_{k}^n$ into $\mathcal{BP}_{l}^{n}$, which is again an immediate corollary of Proposition 3.3 (using $K_2 = D_n \in \mathcal{BP}_{l}^{n}$):

**Proposition 3.5.** If $K \in \mathcal{BP}_{k}^n$ then the star-body $L$ defined by $\rho_L = \rho_{K}^{k/l}$ satisfies $L \in \mathcal{BP}_{l}^{n}$ for $1 \leq k \leq l \leq n - 1$.

We prefer to give another proof of this statement, one which does not rely on Proposition 6.1.

**Proof.** Assume that $K \in \mathcal{BP}_{k}^n$, so that $\rho_{K}^k = R_{n-k}(g_K)$ and $g_K \in C_+^\infty(G(n, n - k))$, and define the star-body $L$ by $\rho_L = \rho_{K}^{k/l}$. For $\theta \in S^{n-1}$ and $a = k, l$ denote by $\mu_{\theta}^{a}$ the Haar probability measure on $G_{\theta}(n, n - a)$. For $F \in G(n, n - l)$, denote by $\mu_{\theta}^{F}$ the Haar probability measure on $G_{F}(n, n - k)$. Then:

$$\rho_{L}^k(\theta) = \rho_{K}^k(\theta) = \int_{G_{\theta}(n, n - k)} g_K(E) d\mu_{\theta}^{k}(E) =$$

$$= \int_{G_{\theta}(n, n - l)} \int_{G_{F}(n, n - k)} g_K(E) d\mu_{\theta}^{k}(E) d\mu_{\theta}^{F}(E).$$

The last transition is justified by the fact that the probability measure $d\mu_{\theta}^{k}(E) d\mu_{\theta}^{F}(E)$ on $G_{\theta}(n, n - k)$ is invariant under orthogonal rotations in $O(n)$ which preserve $\theta$, and
therefore coincides with $d\mu_k^L(E)$, the Haar probability measure on $G_{\theta}(n, n - k)$. Defining $g_L \in C^+_G(G(n, n - l))$ by $g_L(F) = \int_{G_F(n, n - k)} g(E) d\mu_k^L(E)$ for $F \in G(n, n - l)$, we see that:

$$\rho^L_k(\theta) = R^*_k(g_L)(\theta).$$

Together with Remark 3.1, this concludes the proof. \hfill \Box

The Ellipsoid Corollary from the Introduction should now be clear. We repeat it here for convenience:

**Corollary 3.6.** For any $1 \leq k \leq n - 1$ and $k$ ellipsoids $\{E_i\}_{i=1}^k$ in $\mathbb{R}^n$, define the body $L$ by:

$$\rho_L = \rho_{E_1} \cdots \rho_{E_k},$$

and let $k \leq l \leq n - 1$. Then there exists a sequence of star-bodies $\{L_i\}$ which tends to $L$ in the radial metric and satisfies:

$$\rho_{L_i} = \rho_{E_{i,1}}^l + \cdots + \rho_{E_{i,m_i}}^l,$$

where $\{E_{i,j}\}$ are ellipsoids.

**Proof.** The body $L_2$ defined by $\rho_{L_2}^k = \rho_L$ is in $\mathcal{BP}_k^n$ by Proposition 3.3 (applied successively to the ellipsoids $\{E_i\}$, which are in $\mathcal{BP}_1^n$). For $l > k$, Proposition 3.5 implies that the body $L_3$ defined by $\rho_{L_3}^l = \rho_L^k = \rho_L$ is in $\mathcal{BP}_l^n$, otherwise this is trivial. Using Grinberg and Zhang’s characterization of $\mathcal{BP}_l^n$ (Theorem 2.1), the claim is established. \hfill \Box

Incidentally, Proposition 3.3 also enables us to give the following strengthened version of Theorem 2.1:

**Corollary 3.7.** A star-body $K$ is a $k$-Busemann-Petty body iff it is the limit of $\{K_i\}$ in the radial metric, where each $K_i$ is of the following form:

$$\rho_{K_i}^k = \rho_{E_{i,1}} \cdots \rho_{E_{i,k}} + \cdots + \rho_{E_{m_i,1}} \cdots \rho_{E_{m_i,k}},$$

where $\{E_{i,j}\}$ are ellipsoids.

**Proof.** Obviously this representation generalizes the one given by Grinberg and Zhang in Theorem 2.1, so it is enough to show the “if” part. But this follows from the closure of $\mathcal{BP}_k^n$ under limit in the radial metric, $k$-radial sums, and Proposition 3.3 (which as above shows that the body $L$ defined by $\rho_L^k = \rho_{E_1} \cdots \rho_{E_k}$ is in $\mathcal{BP}_k^n$). \hfill \Box

For completeness, we conclude our investigation of the structure of $\mathcal{BP}_k^n$ with the following result of Grinberg and Zhang from [16]. Their argument is the same one used by Goodey and Weil for intersection bodies ($\mathcal{BP}_1^n$), and is an immediate corollary of Theorem 2.1.

**Corollary 3.8 (Grinberg and Zhang).** If $K \in \mathcal{BP}_k^n$ then any $m$-dimensional central section $L$ of $K$ (for $m > k$) satisfies $L \in \mathcal{BP}_k^n$.

**Proof.** Since and central section of an ellipsoid is again an ellipsoid, the claim follows immediately from Theorem 2.1. \hfill \Box
We now turn to prove the Structure Theorem from the Introduction for $\mathcal{I}^n_k$. As will be evident, the techniques involved are totally different from those which were used for $\mathcal{BP}^n_k$.

The only point of similarity is Lemma 3.10 below. We denote by $\mathcal{I}^n_{k,\infty}$ the class of infinitely smooth $k$-intersection bodies in $\mathbb{R}^n$. As mentioned in Section 2, this implies for $K \in \mathcal{I}^n_{k,\infty}$ that $\|\cdot\|_k^{-k}, (\|\cdot\|_k^{-k})^* \in C^\infty(\mathbb{R}^n \setminus \{0\})$. We begin with the following useful lemma:

**Lemma 3.9.** For any $p \in (-n, 0), g \in C^\infty(S^{n-1})$ and $\mu \in \mathcal{M}(O(n)), E_p^\wedge(\mu \ast g) = \mu \ast E_p^\wedge(g)$ as functions on $\mathbb{R}^n \setminus \{0\}$.

**Proof.** First, let us extend the definition of $\mu \ast f$ to any function $f \in C(\mathbb{R}^n)$, as follows: $(\mu \ast f)(x) = \int_{O(n)} f(u(x))d\mu(u)$ for every $x \in \mathbb{R}^n$. Next, notice that for a test function $\phi$, $(\mu \ast \phi)^\wedge = \mu \ast \hat{\phi}$. Indeed, when $\mu$ is a delta function at $u \in O(n), (\phi(u(\cdot)))^\wedge(x) = \hat{\phi}(u(x))$ because the Fourier transform commutes with the action of $O(n)$. And for a general $\mu \in \mathcal{M}(O(n))$, by Fubini’s Theorem:

$$
(\mu \ast \phi)^\wedge(x) = \int_{\mathbb{R}^n} \int_{O(n)} \phi(u(y))d\mu(u)\exp(-i \langle y, x \rangle)dy
$$

$$
= \int_{O(n)} \int_{\mathbb{R}^n} \phi(u(y))\exp(-i \langle y, x \rangle)dyd\mu(u)
$$

$$
= \int_{O(n)} (\phi(u(\cdot)))^\wedge(x)d\mu(u) = \int_{O(n)} \hat{\phi}(u(x))d\mu(u) = \mu \ast \hat{\phi}.
$$

Since $g, \mu \ast g \in C^\infty(S^{n-1})$, it follows that $E_p^\wedge(\mu \ast g), \mu \ast E_p^\wedge(g) \in C^\infty(\mathbb{R}^n \setminus \{0\})$, and for any test function $\phi$:

$$
\langle E_p^\wedge(\mu \ast g), \phi \rangle = \langle E_p(\mu \ast g), \hat{\phi} \rangle = \langle \mu \ast E_p(g), \hat{\phi} \rangle = \langle E_p(g), \mu^{-1} \ast \hat{\phi} \rangle
$$

$$
= \langle E_p(g), (\mu^{-1} \ast \phi)^\wedge \rangle = \langle E_p^\wedge(g), \mu^{-1} \ast \phi \rangle = \langle \mu \ast E_p^\wedge(g), \phi \rangle.
$$

Therefore $E_p^\wedge(\mu \ast g) = \mu \ast E_p^\wedge(g)$ as functions. \qed

**Lemma 3.10.** $\mathcal{I}^n_{k,\infty}$ is dense in $\mathcal{I}^n_k$.

**Proof.** Let $K \in \mathcal{I}^n_k$, and let $\mu \in \mathcal{M}_+(S^{n-1})$ be the measure from Proposition 2.7 satisfying for every $f \in C^\infty(S^{n-1})$:

$$
\int_{S^{n-1}} f(\theta)\rho^k_K(\theta)d\sigma(\theta) = \int_{S^{n-1}} E^\wedge_{n+k}(f)(\theta)d\mu(\theta).
$$

Let $\{u_i\} \subset C^\infty(O(n))$ be an approximate identity as in Lemma 2.3, and let $K_i$ be the star-body for which $\rho^k_{K_i} = u_i \ast \rho^k_K$. Then by Lemma 2.3, $\{K_i\}$ is a sequence of infinitely smooth star-bodies which tend to $K$ in the radial metric. It remains to check that each $K_i$ is a $k$-intersection body. Indeed, using the notations of Section 2 and Lemma 3.9, for any $f \in C^\infty(S^{n-1})$:

$$
\langle f, \rho^k_{K_i} \rangle = \langle f, u_i \ast \rho^k_K \rangle = \langle u_i^{-1} \ast f, \rho^k_K \rangle = \langle E^\wedge_{n+k}(u_i^{-1} \ast f), \mu \rangle
$$

$$
= \langle u_i^{-1} \ast E^\wedge_{n+k}(f), \mu \rangle = \langle E^\wedge_{n+k}(f), u_i \ast \mu \rangle.
$$

Since $u_i \ast \mu \in C^\infty_+(S^{n-1})$, again by Proposition 2.7 this implies that $K_i \in \mathcal{I}^n_k$. \qed
Remark 3.2. By the Lemma and the closure of $I_k^n$ (for any $k = 1, \ldots, n-1$) under limit in the radial metric, it is enough to prove all the remaining items for the classes $I_k^{n, \infty}$.

For the next fundamental proposition, we will need the following observation. It is classical that for two test functions $\phi_1, \phi_2$, $(\phi_1 \phi_2)^\wedge = \hat{\phi}_1 \ast \hat{\phi}_1$ where $\ast$ denotes the standard convolution on $\mathbb{R}^n$. In general, the convolution of two distributions does not exist. Nevertheless, when the two distributions $f_1, f_2$ are locally integrable homogeneous functions with the right degrees, their convolution may be defined as usual. Assume that $f_i$ is even homogeneous of degree $-n + p_i$ for $p_i > 0$ and that $p_1 + p_2 < n$. Since $f_i$ are locally integrable and at infinity their product decays faster than $|x|^{-n}$, the following integral converges for $x \in \mathbb{R}^n \setminus \{0\}$:

\begin{equation}
(3.1) \quad f_1 \ast f_2(x) = \int f_1(x - y)f_2(y)dy.
\end{equation}

It is easy to check that with this definition, $f_1 \ast f_2$ is homogeneous of degree $-n + p_1 + p_2$, hence again locally integrable. Now assume in addition that $f_i$ are infinitely smooth functions on $\mathbb{R}^n \setminus \{0\}$, and therefore so are $\hat{f}_i$. We claim that as distributions $(f_1 \ast f_2)^\wedge = \hat{f}_1 \hat{f}_2$. To see this, we define the product and convolution of an even distribution $f$ with an even test-function $\phi$, as the distributions denoted $\phi f$ and $\phi \ast f$, respectively, satisfying for any test function $\varphi$ that:

\begin{equation}
\langle \phi f, \varphi \rangle = \langle f, \phi \varphi \rangle \quad \text{and} \quad \langle \phi \ast f, \varphi \rangle = \langle f, \phi \ast \varphi \rangle.
\end{equation}

When $f$ is a locally integrable function, it is clear that $\hat{\phi} f$ and $\phi \ast f$ as distributions coincide with the usual product and convolution as functions. The same reasoning shows that when $f_1, f_2$ are locally integrable even functions such that $f_1 f_2$ is integrable at infinity (as before the definition in (3.1)), we have:

\begin{equation}
(3.2) \quad \langle f_1 \ast f_2, \phi \rangle = \langle f_1, \phi \ast f_2 \rangle,
\end{equation}

where the action $\langle \cdot, \cdot \rangle$ is interpreted here and henceforth as integration in $\mathbb{R}^n$. Similarly, when $f_1 f_2$ is locally integrable, we have:

\begin{equation}
(3.3) \quad \langle f_1 f_2, \phi \rangle = \langle f_1, \phi f_2 \rangle.
\end{equation}

With the above definitions, we see that $(\phi \ast f)^\wedge = \hat{\phi} \hat{f}$ because for any test function $\varphi$:

\begin{equation}
(3.4) \quad \langle \phi \ast f, \varphi \rangle = \langle f, \phi \ast \varphi \rangle = \langle \hat{\phi} \hat{f}, \varphi \rangle.
\end{equation}

Now when $f, g$ are two locally integrable infinitely smooth functions on $\mathbb{R}^n \setminus \{0\}$, such that $\hat{f} g$ is locally integrable, it is easy to see that we may replace $\varphi$ in (3.4) with $g$. The reason is that we may weakly approximate $g$ with test functions $g_i$ such that $\int h g_i \to \int h g$ and $\int h g_i \to \int h \hat{g}$, for any locally integrable continuous function $h$ on $\mathbb{R}^n \setminus \{0\}$ such that $\int h g$ exists. For instance, we may use $g_i = (g \ast \delta_i) \delta_i$, where $\delta_i$ are Gaussians tending to a delta-function at 0; by (3.4) it is clear that $\hat{g}_i = (\hat{g} \delta_i) \ast \delta_i$, which weakly tends to $\hat{g}$ (by testing against a test-function). We summarize this by writing:

\begin{equation}
(3.5) \quad \langle \phi \ast f, g \rangle = \langle \hat{\phi} \hat{f}, g \rangle.
\end{equation}
Combining (3.2), (3.3) and (3.5) and using the fact that \(f_1, \hat{f}_1, \hat{f}_1 \hat{f}_2\) are infinitely smooth and locally integrable, we see that for any even test function \(\phi\):

\[
\langle (f_1 * f_2)^\wedge, \phi \rangle = \langle f_1 * f_2, \hat{\phi} \rangle = \langle f_1, \hat{\phi} * f_2 \rangle = \langle \hat{f}_1, \phi f_2 \rangle = \langle \hat{f}_1 \hat{f}_2, \phi \rangle.
\]

This proves that under the above conditions:

\[
(f_1 * f_2)^\wedge = \hat{f}_1 \hat{f}_2.
\]

**Remark 3.3.** Note that the homogeneity of \(f_1, f_2\) was not used, we only needed the appropriate asymptotic behaviour at 0 and infinity. Using the homogeneity, a different approach to derive (3.6) was suggested to us by A. Koldobsky, by applying [13, Lemma 1]. With this approach, the smoothness assumptions on \(f_1, f_2\) may be omitted, and (3.6) is understood as equality between distributions.

Using this notion of convolution, we can now show the following basic property of \(k\)-intersection bodies, and immediately deduce (3a), (3b) and (3c) from the Structure Theorem in the Introduction. The following was also recently noticed independently by Koldobsky (but not published):

**Proposition 3.11.** Let \(K_1 \in T_{k_1}^n\) and \(K_2 \in T_{k_2}^n\) for \(k_1, k_2 \geq 1\) such that \(l = k_1 + k_2 \leq n - 1\). Then the star-body \(L\) defined by \(\rho_L = \rho_{k_1}^{k_1} \rho_{k_2}^{k_2}\) satisfies \(L \in T_l^n\).

**Proof.** First, assume that \(K_i \in T_{k_i}^{n, \infty}\) for \(i = 1, 2\), so that \((||\cdot||^{-k_i}_K)^\wedge \in C_+^\infty(\mathbb{R}^n \setminus \{0\})\) and is homogeneous of degree \(-n + k_i\). Since \(l < n\) the convolution \((||\cdot||^{-k_1}_K)^\wedge * (||\cdot||^{-k_2}_K)^\wedge\) as distributions is well defined (as explained above). Therefore:

\[
(||\cdot||^{-l}_L)^\wedge = (||\cdot||^{-k_1}_K ||\cdot||^{-k_2}_K)^\wedge = (||\cdot||^{-k_1}_K)^\wedge * (||\cdot||^{-k_2}_K)^\wedge \geq 0,
\]

as a function on \(\mathbb{R}^n \setminus \{0\}\), which implies that \(L \in T_l^n\). The general case, when \(K_i \in T_{k_i}^n\) without any smoothness assumptions, follows from Remark 3.2 in the same manner as in the proof of Proposition 3.3.

Applying Proposition 3.11 with \(K_1 = K_2\), we have:

**Corollary 3.12.** \(T_{k_1}^n \cap T_{k_2}^n \subset T_{k_1+k_2}^n\) for \(k_1, k_2 \geq 1\) such that \(k_1 + k_2 \leq n - 1\).

By successively applying Corollary 3.12, we see that \(T_{k}^n \subset T_l^n\) if \(k\) divides \(l\). As for the class \(BP\), the question whether \(T_{k}^n \subset T_l^n\) for general \(1 \leq k < l \leq n - 1\) remains open.

Nevertheless, we are able to show again the following ”non-linear” embedding of \(T_{k}^n\) into \(T_l^n\), which is again an immediate corollary of Proposition 3.11 (using \(K_2 = D_n \in T_{l-n}^n\)):

**Corollary 3.13.** If \(K \in T_{k}^n\) then the star-body \(L\) defined by \(\rho_L = \rho_{k}^{k/l}\) satisfies \(L \in T_l^n\) for \(1 \leq k \leq l \leq n - 1\).

We conclude this section with our last observation:

**Proposition 3.14.** If \(K \in T_{k}^n\) then any \(m\)-dimensional central section \(L\) of \(K\) (for \(m > k\)) satisfies \(L \in T_{k}^m\).
Proof. Let $K$ be a star-body in $\mathbb{R}^n$, fix $k \in \{1, \ldots, n-2\}$, and let $H \in G(n, m)$ for $m > k$. In view of Theorem 2.5, we have to show that as distributions:

\[
(\|\cdot\|^{-k}_K)^{\wedge} \geq 0 \implies (\|\cdot\|^{-k}_K|_H)^{\wedge} = (\|\cdot\|^{-k}_{K\cap H})^{\wedge} \geq 0.
\]

This becomes intuitively clear, after noticing that for a test function $\phi$:

\[
(\phi|_H)^{\wedge}(u) = \int_{u+H^\perp} \hat{\phi}(y)dy.
\]

Nevertheless, for a more general function $f = \|\cdot\|^{-k}_K$ such that $\hat{f} \geq 0$ as a distribution, we will need a somewhat different proof. Note that since $m > k$, $f$ is locally integrable on any affine translate $z + H$, and that for any test function $\phi_H$ on $H$, $\int_H \hat{f}(y+z)\phi(y)dy$ is continuous w.r.t. $z \in H^\perp$. Now let $\phi_H$ be any non-negative test function on $H$. For $\epsilon > 0$, denote by $\varphi_{H,\epsilon}$ the (positive) Gaussian function on $H^\perp$ such that $(\varphi_{H,\epsilon})^{\wedge}$ is the density function of a standard Gaussian variable on $H^\perp$ with covariance matrix $\epsilon I_{H^\perp}$. For $y \in H$ and $z \in H^\perp$, define $\phi_\epsilon(y+z) = \phi_H(y)\varphi_{H,\epsilon}(z)$. Clearly $\phi_\epsilon$ is a test function on $\mathbb{R}^n$, $\phi_\epsilon \geq 0$, and $(\phi_\epsilon)^{\wedge}(y+z) = (\phi_H)^{\wedge}(y)(\varphi_{H,\epsilon})^{\wedge}(z)$. We therefore have:

\[
\langle (f|_H)^{\wedge}, \phi_H \rangle = \langle f|_H, (\phi_H)^{\wedge} \rangle = \int_H f(y)(\phi_H)^{\wedge}(y)dy
\]

\[
= \lim_{\epsilon \to 0} \int_{H^\perp} (\varphi_{H,\epsilon})^{\wedge}(z) \int_H f(y+z)(\phi_H)^{\wedge}(y)dy \, dz
\]

\[
= \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} f(x)(\phi_\epsilon)^{\wedge}(x)dx
\]

\[
= \lim_{\epsilon \to 0} \langle f, (\phi_\epsilon)^{\wedge} \rangle = \lim_{\epsilon \to 0} \langle \hat{f}, \phi_\epsilon \rangle \geq 0.
\]

Since $\phi_H \geq 0$ was arbitrary, it follows that $(f|_H)^{\wedge} \geq 0$. 

\[\square\]

4. The connection between Radon and Fourier Transforms

We have seen that although the classes $\mathcal{BP}_k^n$ and $\mathcal{I}_k^n$ share the exact same structure and easily verify that $\mathcal{BP}_k^n \subset \mathcal{I}_k^n$, they are defined and handled using very different notions: Radon and Fourier transforms, respectively. The aim of this section is to establish a common ground that will enable to attack the question of whether $\mathcal{BP}_k^n = \mathcal{I}_k^n$ from a unified point of view. Since $\mathcal{BP}_k^n \subset \mathcal{I}_k^n$, it seems natural that this common ground will involve the language of Radon transforms, so we will have to translate the action of the Fourier transform to this language.

We will use the following notation. If $\mu \in \mathcal{M}(G(n, n-m))$, we denote by $\mu^\perp \in \mathcal{M}(G(n, n-m))$ the measure defined by $\mu^\perp(A) = \mu(A^\perp)$ for any Borel set $A \subset G(n, m)$, where $A^\perp = \{E^\perp | E \in A\}$. Note that the operation $\mu \mapsto \mu^\perp$ is dual to the operator $I : C(G(n, m)) \to C(G(n, n-m))$ defined in the Introduction, in the sense that $\langle \mu, I(f) \rangle_{G(n, n-m)} = \langle \mu^\perp, f \rangle_{G(n, m)}$. We recall that $I(f)(E) = f^\perp(E) = f(E^\perp)$ for any $E \in G(n, n-m)$. We therefore extend $I$ to an operator $I : \mathcal{M}(G(n, m)) \to \mathcal{M}(G(n, n-m))$, defined as $I(\mu) = \mu^\perp$, and by abuse of notation we say that $I$ is self-dual.
Theorem 2.6 in Section 2 was the first example relating the Radon and Fourier transforms. Using operator notations, this may be stated as:

\[(4.1) \quad R_{n-k} \circ E^\wedge_{-k} = c(n, k)I \circ R_k ,\]

as operators from \(C^\infty(S^{n-1})\) to \(C^\infty(G(n, n-k))\). In view of the remark immediately after Theorem 2.6, a generalization of (4.1) is given by the Spherical Parseval identity, which in the formulation of Remark 2.1, shows that \(E^\wedge_{-k}\) is a self-adjoint operator on \(C^\infty_c(S^{n-1})\):

\[(4.2) \quad (E^\wedge_{-k})^* = E^\wedge_{-k}.\]

Passing to the dual in (4.1) and using (4.2), we immediately have:

\[(4.3) \quad E^\wedge_{-k} \circ R^*_{n-k} = c(n, k)R^*_k \circ I ,\]

as operators on certain spaces. We formulate this more carefully in the next Proposition:

**Proposition 4.1.** Let \(f \in C^\infty_c(S^{n-1})\), and assume that \(f = R^*_{n-m}(d\mu)\) as measures in \(M(S^{n-1})\), for some measure \(\mu \in M(G(n, n-m))\). Then \(E^\wedge_{m}(c(n, m)R^*_m(d\mu^\perp))\) as measures in \(M(S^{n-1})\), where \(c(n, m)\) is the constant from Lemma 2.4.

**Proof.** Let \(g \in C^\infty_c(S^{n-1})\) be arbitrary. Then by the Spherical Parseval identity and Theorem 2.6:

\[
\begin{align*}
\int_{S^{n-1}} E^\wedge_{-m}(f)(\theta)g(\theta)d\sigma(\theta) &= \int_{S^{n-1}} f(\theta)E^\wedge_{m}(g)(\theta)d\sigma(\theta) \\
&= \int_{S^{n-1}} R^*_{n-m}(d\mu)(\theta)E^\wedge_{m}(g)(\theta)d\sigma(\theta) = \int_{G(n, n-m)} R^*_{n-m}(E^\wedge_{-m}(g))(F)d\mu(F) \\
&= c(n, m) \int_{G(n, n-m)} R^*_m(g)(F^\perp)d\mu(F) = c(n, m) \int_{G(n, m)} R_m(g)(F)d\mu^\perp(F) \\
&= c(n, m) \int_{S^{n-1}} R^*_m(d\mu^\perp)(\theta)g(\theta)d\sigma(\theta).
\end{align*}
\]

Since \(C^\infty_c(S^{n-1})\) is dense in \(C_c(S^{n-1})\) in the maximum norm, the proposition follows. \(\square\)

In the context of star-bodies, the following is an immediate corollary of Proposition 4.1:

**Corollary 4.2.** Let \(K\) be an infinitely smooth star-body in \(\mathbb{R}^n\). Then for a measure \(\mu \in M(G(n, n-k))\):

\[
\|\cdot\|\!^k_K = R^*_{n-k}(d\mu) \text{ iff } (\|\cdot\|\!^k_K)^\wedge = c(n, k)R^*_k(d\mu^\perp),
\]

where \(c(n, k)\) is the constant from Lemma 2.4, and the equalities are understood as equalities between measures in \(M(S^{n-1})\).

**Proof.** The "only if" part follows immediately from Proposition 4.1 with \(m = k\) and \(f = \|\cdot\|\!^k_K\). The "if" part follows by applying Proposition 4.1 with \(m = n-k\) and \(f = (\|\cdot\|\!^k_K)^\wedge\), and using the fact that \(E^\wedge_{n-k}(f) = (2\pi)^n\|\cdot\|\!^k_K\) and that the constants \(c(n, k)\) from Lemma 2.4 satisfy \(c(n, k)c(n, n-k) = (2\pi)^n\). \(\square\)

Proposition 4.1 has several interesting consequences. The first one is:
Theorem 4.3. Let $n > 1$ and fix $1 \leq k \leq n - 1$. Then:
\[ \mathcal{BP}_k^n = \mathcal{I}_k^n \iff \mathcal{BP}_{n-k}^n = \mathcal{I}_{n-k}^n. \]

Proof. Assume that $\mathcal{BP}_{n-k}^n = \mathcal{I}_{n-k}^n$, and let $K \in \mathcal{I}_{n-k}^\infty$. In view of Lemma 3.10, the fact that $\mathcal{BP}_k^n$ is closed under limit in the radial metric, and Corollary 2.2, it is enough to show that $K \in \mathcal{BP}_k^n$. Since $(\|\cdot\|_K^{-k})^\wedge \geq 0$ by Theorem 2.5, we may define the infinitely smooth star-body $L$ as the body for which $\|\cdot\|_{L}^{-n+k} = (\|\cdot\|_K^{-k})^\wedge$. Therefore $(\|\cdot\|_{L}^{-n+k})^\wedge = (2\pi)^n \|\cdot\|_K^{-k} \geq 0$, hence $L \in \mathcal{I}_{n-k}^n$. It follows from our assumption that $L \in \mathcal{BP}_{n-k}^n$, so there exists a non-negative measure $\mu \in \mathcal{M}_+(G(n,k))$ so that $(\|\cdot\|_{L}^{-n+k})^\wedge = \|\cdot\|_{L}^{-n+k} = R_k^n(\mu)$. By Corollary 4.2, this implies that $\|\cdot\|_{K}^{-k} = c(n,k)R_k^n(\mu^{-1})$. Therefore $K \in \mathcal{BP}_k^n$, which concludes the proof. \[ \square \]

Another immediate consequence of Proposition 4.1 is another elementary proof of:

Corollary 4.4.
\[ \mathcal{BP}_k^n \subset \mathcal{I}_k^n. \]

Proof. Let $K \in \mathcal{BP}_k^n$ so $\|\cdot\|_{K}^{-k} = R_{n-k}^n(\mu)$ for some non-negative Borel measure $\mu \in \mathcal{M}_+(G(n,n-k))$. By Corollary 4.2 of Proposition 4.1, it follows that $(\|\cdot\|_{K}^{-k})^\wedge = c(n,k)R_k^n(\mu^{-1})$, implying that $(\|\cdot\|_{K}^{-k})^\wedge \geq 0$, and hence $K \in \mathcal{I}_k^n$. By Lemma 3.1, and the fact that $\mathcal{I}_k^n$ is closed under limit in the radial metric, this concludes the proof. \[ \square \]

Applying Proposition 4.1 to the function $f = 0$, once for $m = k$ and once for $m = n - k$, we also immediately deduce the following useful:

Proposition 4.5.
\[ \text{Ker} R_{n-k}^n = \text{Ker} R_k^n \circ I. \]

This is equivalent by a standard duality argument to the following Proposition, which may be deduced directly from Theorem 2.6:

Proposition 4.6.
\[ \text{Im} R_{n-k}^n = \text{Im} I \circ R_k^n. \]

We conclude this section by introducing a family of very natural operators acting on $C(G(n,k))$ to itself, and showing a few nice properties which they share. Denote by $V_k : C(G(n,k)) \to C(G(n,k))$ the operator defined as $V_k = I \circ R_{n-k} \circ R_k^n$.

Proposition 4.7. $V_k$ is self-adjoint.

Proof. It is actually not hard to show this directly, just by using double-integration as in Section 3. Nevertheless, we prefer to use Proposition 4.1. Let $f, g \in C^\infty(G(n,n-k))$. Then by Proposition 4.1, the spherical Parseval identity and Proposition 4.1 again, we have:
\[ \langle V_{n-k}(f), g \rangle_{G(n,n-k)} = \langle R_{n-k}^n(f), (I \circ R_k^n)^* (g) \rangle = c(n,k)^{-1} \langle R_{n-k}^n(f), (E_{-k}^n \circ R_{n-k}^n)(g) \rangle = c(n,k)^{-1} \langle (E_{-k}^n \circ R_{n-k}^n)(f), R_{n-k}^n(g) \rangle = \langle (I \circ R_k^n)^* (f), R_{n-k}^n(g) \rangle = \langle f, V_{n-k}(g) \rangle_{G(n,n-k)} \]
Since $C^\infty(G(n,n-k))$ is dense in $C(G(n,n-k))$ in the maximum norm, and the operators $R_{n-k}^*$ and $R_k$, and hence $V_{n-k}$, are continuous w.r.t. this norm, it follows that the same holds for any $f,g \in C(G(n,n-k))$.

\begin{proposition}
\[ V_{n-k} = I \circ V_k \circ I. \]
\end{proposition}

\textbf{Proof.} This time we give the proof in operator style notations. The formal details are filled in exactly the same manner as above. Using the definition of $V_k$, and the identities (4.3) and (4.1), we have:

\[ I \circ V_k \circ I = R_{n-k} \circ R_k^* \circ I = c(n,k)^{-1} R_{n-k} \circ E_k^* \circ R_{n-k}^* = I \circ R_k \circ R_{n-k}^* = V_{n-k}. \]

\qed

It is known (e.g. [9]) that for $1 < k < n - 1$, even if we restrict the operators $R_m$ to infinitely smooth functions, $KerR_k^* \neq \{0\}$ and $\overline{ImR_{n-k}} \neq C^\infty(G(n,n-k))$, and therefore $V_k$ is neither injective nor surjective onto a dense set for those values of $k$. Since $\overline{ImR_k^*} = C_\subseteq(S^{n-1})$ and $KerR_{n-k} = \{0\}$, it follows that $KerV_k = KerR_k^*$ and $\overline{ImV_k} = \overline{ImI \circ R_{n-k}} = \overline{ImR_k^*}$ (by Proposition 4.6). A standard duality argument shows that $\overline{ImR_k^*}$ is orthogonal to $KerR_k^*$ (as measures acting on continuous functions, and therefore as functions when $R_k^*$ is restricted to $C(G(n,k))$), and therefore we may consider $V_k$ as an operator from $\overline{ImR_k^*}$ to $\overline{ImR_k}$, which is injective and surjective onto a dense set. A natural question for Integral Geometers would be to find a nice inversion formula for $V_k$. Note that by a standard double-integral argument, the operator $R_k^* \circ I \circ R_{n-k} : S^{n-1} \rightarrow S^{n-1}$ is exactly the usual Spherical Radon transform $R$ (for every $k$), and under the standard identification between $G(n,n-1)$, $G(n,1)$ and $S^{n-1}$, so are $V_1$ and $V_{n-1}$.

5. Equivalent formulations of $\mathcal{BP}_k^n = T_k^n$

In this section we use the results and techniques of the previous sections together with basic tools from Functional Analysis to derive equivalent formulations of the natural conjecture that $\mathcal{BP}_k^n = T_k^n$. As mentioned in the Introduction, the relevance of this conjecture to Convex Geometry stems from the generalized $k$-codimensional Busemann-Petty problem. It was shown in [36] that the answer to this problem is positive iff every convex body in $\mathbb{R}^n$ is in $\mathcal{BP}_k^n$, and this was shown to be false ([4],[21]) for $k < n - 3$, but the cases of $k = n - 3$ and $k = n - 2$ remain open. The analogous question for $T_k^n$ turned out to be easier using the analytic tools provided by the Fourier transform, and it was shown by Koldobsky in [20] that $T_k^n$ contains all $n$-dimensional convex bodies iff $k \geq n - 3$. Hence, a positive answer to whether $\mathcal{BP}_k^n = T_k^n$ would imply a positive answer to the generalized $k$-codimensional Busemann-Petty problem, for $k \geq n - 3$. The equivalent formulations derived in this section indicate that the $\mathcal{BP}_k^n = T_k^n$ question is connected and equivalent to very natural questions in Integral Geometry.

Before we start, we would like to give an intuitive equivalent formulation to $\mathcal{BP}_k^n = T_k^n$. By Grinberg and Zhang’s characterization (Theorem 2.1), $\mathcal{BP}_k^n$ is exactly the class of star-bodies generated from the Euclidean Ball $D_n$ by means of full-rank linear transformations, $k$-radial sums, and limit in the radial metric. Loosely speaking, we say that "modulo these
operations", $D_n$ is the only member of $\mathcal{BP}_n$. Since $S^n_1$ is closed under these operations as well, we can ask whether "modulo these operations" $D_n$ is the only star-body such $(\|\cdot\|^{-k}_{D_n})^\wedge \geq 0$. In terms of functions on the sphere, this is equivalent to asking whether "modulo these operations", the only function $f \in C_c^\infty(S^{n-1})$ such that $f \geq 0$ and $E^\infty_{n,k}(f) \geq 0$ is the constant function $f = 1$ (note that we may restrict our attention to infinitely smooth functions because of Lemma 3.10). This formulation transforms the problem to the language of Fourier transforms. As opposed to this, our other formulations in this section will use the language of the Radon transforms and Integral Geometry.

We will use the following notations. $R_m(C(S^{n-1}))_+$ will denote the non-negative functions in the image of $R_m$ and $R_m(C_+(S^{n-1}))$ will denote the image of $R_m$ acting on the cone $C_+(S^{n-1})$ (which is the same as its image acting on $C_{+,e}(S^{n-1})$). We denote $G = G(n, n - k)$ for short.

It is well known (e.g. [9],[17],[33]) that $R_{n-k} : C_e(S^{n-1}) \to C(G(n, n - k))$ is an injective operator, but it is not onto for $k < n - 1$, and $\text{Im} R_{n-k} \neq C(G(n, n - k))$ for $1 < k < n - 1$. We will restrict our discussion to this range of $k$. It follows by an elementary duality argument, that the image of the dual operator $R^*_n \rightarrow M(G(n, n - k)) \to M_e(S^{n-1})$ is dense in $M_e(S^{n-1})$ in the $w^*$-topology, but $R^*_{n-k}$ is not injective and has a non-trivial kernel. It is known that the dense image in $M_e(S^{n-1})$ contains $C^\infty_e(S^{n-1})$, and in fact an explicit inversion formula was obtained by Koldobsky in [21, Proposition 3] (which is not unique because of the kernel). It follows from Koldobsky’s argument (or from the general results of [9]) that:

**Lemma 5.1.** If $f \in C^\infty_e(S^{n-1})$ then there exists a $g \in C^\infty(G(n, n - k))$ such that $f = R^*_{n-k}(g)$.

It will also be useful to note that:

\[
\text{Ker} R^*_{n-k} = \{ \mu \in M(G(n, n - k)) : \langle \mu, f \rangle = 0 \text{ } \forall f \in \text{Im} R_{n-k} \},
\]

and to recall Propositions 4.5 and 4.6, which show that $\text{Ker} R^*_{n-k} = \text{Ker} R^*_k \circ I$ and $\text{Im} R_{n-k} = \text{Im} I \circ R_k$. The latter immediately implies:

\[
R_{n-k}(C_+(S^{n-1})), I \circ R_k(C_+(S^{n-1})) \subset \frac{R_{n-k}(C(S^{n-1}))}{\text{Ker} R^*_{n-k}}.
\]

It will be useful to consider the quotient space:

\[
M(n, n - k) = M(G(n, n - k))/\text{Ker} R^*_{n-k},
\]

which is the space of bounded linear functionals on the subspace $\text{Im} R_{n-k}$ of $C(G(n, n - k))$. By abuse of notation, we will also think of $R^*_{n-k}$ as an operator from $M(n, n - k)$ to $M_e(S^{n-1})$, and although this does not change its image, it is now injective on $M(n, n - k)$. The same is true for $R^*_k \circ I$, since $\text{Ker} R^*_{n-k} = \text{Ker} R^*_k \circ I$, and we may proceed to interpret $R^*_{n-k}(d\mu)$ and $R^*_k(d\mu^\perp)$ in the usual way for $\mu \in M(n, n - k)$, since these values are the same for the entire co-set $\mu + \text{Ker} R^*_{n-k}$. If $R^*_{n-k}$ were onto $M_e(S^{n-1})$, or even $C^\infty_e(S^{n-1})$, we could proceed by identifying between a star-body $K$ and a signed Borel measure $\mu$ in $M(n, n - k)$, by the correspondence $\|\cdot\|_K^K = R^*_{n-k}(d\mu)$. Unfortunately, the general theory does not guarantee this, and in fact we believe that some star-bodies do not admit such a
representation (although we have not been able to find a reference for this). But as remarked earlier, \( C_\infty^e(S^{n-1}) \) does lie in the image of \( R_{n-k}^* \), and this is enough for our purposes.

Let us now review the definitions of \( \mathcal{BP}_k^n \) and \( T_k^n \). Our original definition required that \( K \in \mathcal{BP}_k^n \) iff \( \rho_K^k = R_{n-k}^*(d\mu) \) for some non-negative measure \( \mu \in \mathcal{M}_+(G(n, n-k)) \). We claim that this is equivalent to requiring that \( \mu \in \mathcal{M}_+(n, n-k) \), since by a version of the Hahn-Banach Theorem ([16, Lemma 4.3]), any non-negative functional on \( \mathcal{M}_+(n, n-k) \) may be extended to a non-negative functional on the entire \( C(G(n, n-k)) \), and the converse is trivially true. Defining \( \mathcal{M}(\mathcal{BP}_k^n) \) as the set of non-negative functionals in \( \mathcal{M}(n, n-k) \):

\[
\mathcal{M}(\mathcal{BP}_k^n) = \mathcal{M}_+(n, n-k),
\]

we see that:

**Lemma 5.2.** Let \( K \) be a star-body in \( \mathbb{R}^n \). Then \( K \in \mathcal{BP}_k^n \) iff \( \rho_K^k = R_{n-k}^*(d\mu) \), for some \( \mu \in \mathcal{M}(\mathcal{BP}_k^n) \).

Let us also define \( \mathcal{M}(I_k^n) \) as:

\[
\mathcal{M}(I_k^n) = \left\{ \mu \in \mathcal{M}(n, n-k) \mid R_{n-k}^*(d\mu) \geq 0, R_k^*(d\mu^+) \geq 0 \right\},
\]

where "\( \nu \geq 0 \)" means that \( \nu \) is a non-negative measure in \( \mathcal{M}_e(S^{n-1}) \). Using co-set notations, let us also define:

\[
\mathcal{M}^\infty(n, n-k) = \left\{ f + Ker R_{n-k}^* \mid f \in C^\infty(G) \right\},
\]

and denote:

\[
\mathcal{M}^\infty(I_k^n) = \mathcal{M}(I_k^n) \cap \mathcal{M}^\infty(n, n-k),
\]

and

\[
\mathcal{M}_+^\infty(n, n-k) = \mathcal{M}^\infty(\mathcal{BP}_k^n) = \mathcal{M}(\mathcal{BP}_k^n) \cap \mathcal{M}^\infty(n, n-k).
\]

Unfortunately, we cannot give a completely analogous characterization to Lemma 5.2 for \( I_k^n \) and \( \mathcal{M}(I_k^n) \). However, we have the following:

**Lemma 5.3.** Let \( K \) be an infinitely smooth star-body in \( \mathbb{R}^n \). Then \( K \in I_k^n \) iff \( \rho_K^k = R_{n-k}^*(d\mu) \), for some \( \mu \in \mathcal{M}^\infty(I_k^n) \).

*Proof.* We will first prove the "only if" part. Assume that \( K \in I_k^n \). By Lemma 5.1, there exists a signed measure \( \mu \in \mathcal{M}^\infty(n, n-k) \) so that \( \|\cdot\|_{K}^k = R_{n-k}^*(d\mu) \). By Corollary 4.2 of Proposition 4.1, it follows that \( \|\cdot\|_{K}^k \geq 0 \) because \( K \) is a star-body and \( \|\cdot\|_{K}^k \geq 0 \) because \( K \in I_k^n \). It follows that \( R_{n-k}^*(d\mu) \geq 0 \) and \( R_k^*(d\mu^+) \geq 0 \), proving that \( \mu \in \mathcal{M}^\infty(I_k^n) \). The "if" part follows from Corollary 4.2 in exactly the same manner, since \( \|\cdot\|_{K}^k \geq 0 \) implies that \( \mu \in \mathcal{M}^\infty(I_k^n) \) such that \( \|\cdot\|_{K}^k = R_{n-k}^*(d\mu) \). \( \square \)

**Remark 5.1.** It seems that any attempt to prove the "only if" part of the lemma for a general star-body \( K \in I_k^n \) by approximating it with \( K_i \in I_k^n \) will fail. The reason is that we have no way of controlling the norm of the (a-priori signed) measures \( \mu_i \in \mathcal{M}(I_k^n) \) for which \( \rho_K^k = R_{n-k}^*(d\mu_i) \), and therefore it is not guaranteed that \( \mu_i \) will converge to some measure (like in the usual argument which uses the \( w^* \)-compactness of the unit-ball of \( \mathcal{M}(n, n-k) \)). If it were known that the \( \mu_i \) are non-negative (this would follow if \( \mathcal{M}(\mathcal{BP}_k^n) = \mathcal{M}(I_k^n) \)),
it would follow that $\|\mu_i\| = \|R_{n-k}^*(d\mu_i)\|$ (since $R_{n-k}^*(d\mu_i)$ is non-negative), and over the latter term we do have control. The "if" part of the lemma may be proved without any smoothness assumption by the standard approximation argument.

We now see that we have derived alternative definitions of $\mathcal{BP}_k^n$ and $\mathcal{I}_{k}^{n,\infty}$ using a common language of Radon transforms and without using the Fourier transform. Note that even if we could remove the restriction of infinite smoothness from Lemma 5.3, it would not be yet clear that $\mathcal{BP}_k^n = \mathcal{I}_{k}^n$ iff $\mathcal{M}(\mathcal{BP}_k^n) = \mathcal{M}(\mathcal{I}_{k}^n)$, since for a general $\mu \in \mathcal{M}(\mathcal{BP}_k^n)$ or $\mu \in \mathcal{M}(\mathcal{I}_{k}^n)$, $R_{n-k}^*(d\mu)$ may not be a measure with continuous density (and hence cannot equal $\rho_k^*$ for a star-body $K$). We do however have:

**Lemma 5.4.**

$$\mathcal{M}(\mathcal{BP}_k^n) \subset \mathcal{M}(\mathcal{I}_{k}^n)$$

**Proof.** If $\mu \in \mathcal{M}_+(n,n-k)$ then trivially $R_{n-k}^*(d\mu) \geq 0$ and $R_{k}^*(d\mu^\perp) \geq 0$, hence $\mu \in \mathcal{M}(\mathcal{I}_{k}^n)$. Although the proof is trivial, note that underlying this statement are Propositions 4.5 and 4.6 which enabled us to restrict $R_{n-k}^*$ and $R_k^\perp$ to $I$ to $\mathcal{M}(n,n-k)$. $\square$

We may now formulate the main Theorem of this section:

**Theorem 5.5.** Let $n$ and $1 \leq k \leq n-1$ be fixed. Then the following are equivalent:

1. $\mathcal{BP}_k^n = \mathcal{I}_{k}^n$.
2. $\mathcal{M}^\infty(\mathcal{BP}_k^n) = \mathcal{M}^\infty(\mathcal{I}_{k}^n)$.
3. $\mathcal{M}(\mathcal{BP}_k^n) = \mathcal{M}(\mathcal{I}_{k}^n)$.
4. $R_{n-k}(C(S^{n-1}))_+ = R_{n-k}(C_+(S^{n-1})) + I \circ R_k(C_+(S^{n-1}))$.
5. If $\mu + 1 \in \mathcal{M}(\mathcal{BP}_k^n)$ and $\mu \in \mathcal{M}(\mathcal{I}_{k}^n)$, then $\mu \in \mathcal{M}(\mathcal{BP}_k^n)$.
6. There does not exist a measure $\mu \in \mathcal{M}_+(n,n-k)$ such that $R_{n-k}^*(d\mu) \geq 1$ and $R_k^*(d\mu^\perp) \geq 1$ (where "$\nu \geq 1"$ means that $\nu - 1$ is a non-negative measure), and such that:

$$\inf \{ \langle \mu, f \rangle | f \in R_{n-k}(C(S^{n-1}))_+ \text{ and } (1, f) = 1 \} = 0.$$ 

We will show (2) $\Rightarrow$ (1), (1) $\Rightarrow$ (3), (3) $\Leftrightarrow$ (4), (5) $\Rightarrow$ (6) and (6) $\Rightarrow$ (2). Obviously, (3) $\Rightarrow$ (2) and (3) $\Rightarrow$ (5).

**Proof of (2) $\Rightarrow$ (1).** Let $K \in \mathcal{I}_{k}^{n,\infty}$. In view of Lemma 3.10, the fact that $\mathcal{BP}_k^n$ is closed under limit in the radial metric, and Corollary 2.2, it is enough to show that $K \in \mathcal{BP}_k^n$. By Lemma 5.3, $\rho_k^* = R_{n-k}^*(d\mu)$ for some $\mu \in \mathcal{M}(\mathcal{I}_{k}^n)$. By our assumption that $\mathcal{M}(\mathcal{BP}_k^n) = \mathcal{M}(\mathcal{I}_{k}^n)$ and by Lemma 5.2, it follows that $K \in \mathcal{BP}_k^n$ (in fact $K \in \mathcal{BP}_k^{n,\infty}$). $\square$

**Proof of (1) $\Rightarrow$ (3).** In view of Lemma 5.4, it is enough to prove $\mathcal{M}(\mathcal{I}_{k}^n) \subset \mathcal{M}(\mathcal{BP}_k^n)$. Let $\mu \in \mathcal{M}(\mathcal{I}_{k}^n)$, so $R_{n-k}^*(d\mu) \geq 0$ and $R_k^*(d\mu^\perp) \geq 0$. Let $\{u_i\} \subset C^\infty(O(n))$ be an approximate identity as in Lemma 2.3. Let $K_i$ denote the infinitely smooth star-body defined by:

$$\|K_i^{-1} = u_i \ast R_{n-k}^*(\mu) \geq 0$$
(we used $R^*_{n-k}(\mu) \geq 0$ to verify that $K_i$ is indeed a star-body). As in the proof of Lemma 3.1, it is easy to see that $\|\cdot\|_{K_i}^{-k} = R^*_{n-k}(u_i * \mu)$, so by Corollary 4.2 of Proposition 4.1 we have:

$$\|\cdot\|_{K_i}^{-k} = c(n, k) R^*_{k}(u_i * \mu) = R^*_{k}(u_i * \mu) = u_i * R^*_{k}(\mu) \geq 0.$$ 

Hence $K_i \in \mathcal{I}_k^n$, and by our assumption that $\mathcal{BP}_k^n = \mathcal{I}_k^n$, it follows that $K_i \in \mathcal{BP}_k^n$. By Lemma 5.2, this implies that $\|\cdot\|_{K_i}^{-k} = \mathcal{M}(BP_k^n)$, where $\eta_0 \in \mathcal{M}(BP_k^n)$. The injectivity of $R^*_{n-k}$ on $\mathcal{M}(n, n-k)$ implies that $u_i * \mu = \eta_0 \in \mathcal{M}(BP_k^n)$. Lemma 2.3 shows that $u_i * \mu$ tends to $\mu$ in the $w^*$-topology, and since $\mathcal{M}(BP_k^n)$ is obviously closed in this topology, it follows that $\mu \in \mathcal{M}(BP_k^n)$.

For the proof of (3) $\iff$ (4) and for later use, we will need to recall a few classical notions from Functional Analysis (e.g. [2]). A cone $P$ in a Banach space $X$ is a non-empty subset of $X$ such that $x, y \in P$ implies $c_1 x + c_2 y \in P$ for every $c_1, c_2 \geq 0$. The dual cone $P^* \subset X^*$ is defined by $P^* = \{ x^* \in X^* \mid \langle x^*, p \rangle \geq 0 \ \forall p \in P \}$. Therefore $P^*$ is always closed in the $w^*$-topology, and $P^* = (P^*)^*$. It is also easy to check that $P_1 \subset P_2$ implies $P_2^* \subset P_1^*$, $(P_1 + P_2)^* = P_1^* \cap P_2^*$ and $(P_1 \cap P_2)^* = P_1^* + P_2^*$. An immediate consequence of the Hahn-Banach Theorem is that $\overline{P_1} = P_2$ iff $P_1^* = P_2^*$.

Proof of (3) $\iff$ (4). All the sets appearing in (3) and (4) are clearly cones. It remains to show that the cones in both sides of (3) are exactly the dual cones to the ones in both sides of (4). The equivalence then follows by the Hahn-Banach Theorem, as in the last statement of the previous paragraph.

By definition, $\mathcal{M}(BP_k^n)$ is dual to $\overline{R_{n-k}(C(S^{n-1}))}$. The cones:

$$\{ \mu \in \mathcal{M}(n, n-k) \mid R^*_{n-k}(d\mu) \geq 0 \}$$

and

$$\{ \mu \in \mathcal{M}(n, n-k) \mid R^*_{k}(d\mu) \geq 0 \}$$

are immediately seen to be dual to $R_{n-k}(C_+(S^{n-1}))$ and $I \circ R_k(C_+(S^{n-1}))$, respectively. Since $(P_1 + P_2)^* = P_1^* \cap P_2^*$, it follows that:

$$\mathcal{M}(\mathcal{I}_k^n) = \left( \overline{R_{n-k}(C_+(S^{n-1}))} + I \circ R_k(C_+(S^{n-1})) \right)^*.$$ 

This concludes the proof. \qed

Remark 5.2. By (5.2), we have:

$$R_{n-k}(C(S^{n-1}))^+ \subset R_{n-k}(C_+(S^{n-1})) + I \circ R_k(C_+(S^{n-1})).$$

By duality, we see again that:

$$\mathcal{M}(BP_k^n) \subset \mathcal{M}(\mathcal{I}_k^n).$$

Proof of (5) $\Rightarrow$ (6). This follows immediately from the definitions. Assume that (6) is false, so that there exists a measure $\mu \in \mathcal{M}(n, n-k)$ such that $R^*_{n-k}(d\mu) \geq 1$ and $R^*_{k}(d\mu) \geq 1$ and such that (5.3) holds. Define $\mu' = \mu - 1$, and so $\mu' + 1 \in \mathcal{M}(BP_k^n)$, $\mu' \in \mathcal{M}(\mathcal{I}_k^n)$, and (5.3) shows that $\mu'$ is not in $\mathcal{M}(BP_k^n)$. Therefore $\mu'$ is a counterexample to (5). \qed
Proof of (6) ⇒ (2). Assume that (2) is false, so \( \mathcal{M}^\infty(\mathcal{BP}_k^n) \neq \mathcal{M}^\infty(\mathcal{I}_k^n) \). By Lemma 5.4, this means that there exists a measure \( \mu' \in \mathcal{M}^\infty(\mathcal{I}_k^n) \setminus \mathcal{M}(\mathcal{BP}_k^n) \). Since \( \mu' \in \mathcal{M}^\infty(n, n-k) \), we can write \( \mu' = g + \text{Ker} R_{n-k}^* \) with \( g \in C^\infty(G(n, n-k)) \). Assume that \( \min(g) = -C \) where \( C > 0 \), otherwise we would have \( \mu' \in \mathcal{M}(\mathcal{BP}_k^n) \).

Now consider the measure \( \mu_\lambda = (1-\lambda)\mu' + \lambda \in \mathcal{M}^\infty(n, n-k) \) for \( \lambda \in [0, 1] \). Since \( \mathcal{M}(\mathcal{BP}_k^n) \) is convex, contains the measure 1, and is closed in the \( w^* \)-topology, it follows that there exists a \( \lambda_0 \in (0, 1] \) so that \( \mu_\lambda \in \mathcal{M}^\infty(\mathcal{BP}_k^n) \) iﬀ \( \lambda \in [\lambda_0, 1] \). But for \( \lambda_1 = C/(1+C) \) we already see that \( \mu_{\lambda_1} \in \mathcal{M}(\mathcal{BP}_k^n) \), because \( \mu_{\lambda_1} = g_{\lambda_1} + \text{Ker} R_{n-k}^* \) and \( g_{\lambda_1} = 1/(1+C)g + 1-1/(1+C) \in C^\infty_+(G(n, n-k)) \). We conclude that \( \lambda_0 \in (0, 1] \).

Now define \( \mu = \mu_{\lambda_0}/\lambda_0 \in \mathcal{M}(\mathcal{BP}_k^n) \), and notice that \( \mu - 1 = (1-\lambda_0)/\lambda_0 \mu' \in \mathcal{M}^\infty(\mathcal{I}_k^n) \), implying that \( R_{n-k}^*(d\mu) \geq 1 \) and \( R_k^*(d\mu^{-1}) \geq 1 \). It remains to show (5.3). Assume by negation that:

\[
\inf \{ \langle \mu, f \rangle \mid f \in R_{n-k}(C(S^{n-1}))_+ \text{ and } \langle 1, f \rangle = 1 \} = \delta > 0.
\]

But then it is easy to check that for \( \lambda_2 = \lambda_0(1-\delta)/(1-\delta \lambda_0) < \lambda_0 \), \( \langle \mu_{\lambda_2}, f \rangle \geq 0 \) for all \( f \in R_{n-k}(C(S^{n-1}))_+ \), and hence for all \( f \in R_{n-k}(C(S^{n-1}))_+ \). Therefore \( \mu_{\lambda_2} \in \mathcal{M}^\infty(\mathcal{BP}_k^n) \), in contradiction to the definition of \( \lambda_0 \). Therefore (5.3) is shown, concluding the proof.

\[\square\]

Remark 5.3. In formulation (6), it is equivalent to require that \( \mu \in \mathcal{M}_+(n, n-k) \) and also \( \mu \in \mathcal{M}(G(n, n-k)) \) instead of \( \mu \in \mathcal{M}^\infty(n, n-k) \). The equivalence of \( \mu \in \mathcal{M}_+(n, n-k) \) follows since we have not used the fact that \( \mu \in \mathcal{M}^\infty(n, n-k) \) in the proof (by negation) of (5) \( \Rightarrow \) (6). The equivalence of \( \mu \in \mathcal{M}(G(n, n-k)) \) follows by the previously mentioned version of the Hahn-Banach Theorem (which was used to derive Lemma 5.2). This is the formulation which was used in the Introduction.

We proceed to develop several more formulations of the \( \mathcal{BP}_k^n = \mathcal{I}_k^n \) question. Unfortunately, we cannot show an equivalence with the original question, but rather a weak type of implication. We formulate a very natural conjecture, and show that together with a positive answer to the \( \mathcal{BP}_k^n = \mathcal{I}_k^n \) question, the new formulations are implied.

Given an Borel set \( Z \subset G(n, n-k) \), we define the restriction of a measure \( \mu \in \mathcal{M}(G(n, n-k)) \) to \( Z \), denoted \( \mu|_Z \in \mathcal{M}(G(n, n-k)) \), as the measure satisfying \( \mu|_Z(A) = \mu(A \cap Z) \) for any Borel set \( A \subset G(n, n-k) \). We will say that \( \mu \) is supported in a closed set \( Z \), if \( \mu|_Z = 0 \), and define the support of \( \mu \), denoted \( \text{supp}(\mu) \), as the minimal closed set \( Z \) in which \( \mu \) is supported (it is easy to check that this is well-defined). It is also easy to check that:

**Lemma 5.6.** If \( f \in C(G(n, n-k)) \), \( \mu \in \mathcal{M}(G(n, n-k)) \) and \( \text{supp}(\mu) \subset f^{-1}(0) \) then \( \langle \mu, f \rangle = 0 \). Conversely, if \( f \in C_+(G(n, n-k)) \), \( \mu \in \mathcal{M}_+(G(n, n-k)) \) and \( \langle \mu, f \rangle = 0 \), then \( \text{supp}(\mu) \subset f^{-1}(0) \).

We also recall the definition of the Covering Property from the Introduction. A set closed \( Z \subset G(n, n-k) \) is said to satisfy the covering property if:

\[
(5.4) \quad \bigcup_{E \in Z} E \cap S^{n-1} = S^{n-1} \quad \text{and} \quad \bigcup_{E \in Z} E^1 \cap S^{n-1} = S^{n-1}.
\]

Our starting point is formulation (6) in Theorem 5.5, which involves both a function \( f \) and a measure \( \mu \). Note that the requirement that if \( f \in R_{n-k}(C(S^{n-1}))_+ \) and \( \langle 1, f \rangle = 1 \),
then \((\mu, f)\) is bounded away from zero, is stronger than demanding that \(\langle \mu, f \rangle \neq 0\). The motivation for the following discussion stems from the impression that the conditions on \(\mu\), namely that \(\mu \in \mathcal{M}_+(G(n, n-k))\) (following Remark 5.3), \(R^*_{n-k}(d\mu) \geq 1\) and \(R^*_k(d\mu^+) \geq 1\), may be equivalently specified by some condition on the support of \(\mu\). In that case, the condition that \(\langle \mu, f \rangle \neq 0\) becomes a condition on the set \(f^{-1}(0)\). Let us show the following necessary condition on the support of such a \(\mu\) as above:

**Lemma 5.7.** Let \(\mu \in \mathcal{M}_+(G(n, n-k))\) so that \(R^*_{n-k}(d\mu) \geq 1\) and \(R^*_k(d\mu^+) \geq 1\). Then \(\text{supp}(\mu)\) satisfies the covering property.

**Proof.** Denote by \(Z = \text{supp}(\mu)\) and \(\tilde{Z} = \bigcup_{E \in Z} E \cap S^{n-1}\). We will show that if \(\mu \in \mathcal{M}_+(G(n, n-k))\) and \(R^*_{n-k}(d\mu) \geq 1\) then \(\tilde{Z} = S^{n-1}\). The other "half" of the covering property follows similarly from \(R^*_k(d\mu^+) \geq 1\).

Notice that for \(E_1, E_2 \in G(n, n-k)\), the Hausdorff distance between \(E_1 \cap S^{n-1}\) and \(E_2 \cap S^{n-1}\) is equivalent to the distance between \(E_1\) and \(E_2\) in \(G(n, n-k)\). It follows that since \(Z\) is closed, so is \(\tilde{Z}\). Now assume that \(\tilde{Z} \neq S^{n-1}\), so there exists a \(\theta \in S^{n-1}\) and an \(\epsilon > 0\), so that \(\tilde{B} = B_{S^{n-1}}(\theta, \epsilon) \cup B_{S^{n-1}}(-\theta, \epsilon) \subseteq \tilde{Z}^C\). Let \(f \in C_{e,+}(S^{n-1})\) be any non-zero function supported in \(\tilde{B}\). Since \(\tilde{B} \subseteq \tilde{Z}^C\) it follows that \(B = \text{supp}(R_{n-k}(f)) \subset Z^C\), and therefore:

\[
\langle R^*_{n-k}(\mu), f \rangle = \langle \mu, R_{n-k}(f) \rangle = 0.
\]

But on the other hand, since \(R^*_{n-k}(d\mu) \geq 1\) and \(f \in C_{e,+}(S^{n-1})\) is non-zero:

\[
\langle R^*_{n-k}(\mu), f \rangle \geq \langle 1, f \rangle > 0,
\]

a contradiction. \(\Box\)

We conjecture that the covering property is also a sufficient condition in the following sense:

**Covering Property Conjecture.** For any \(n > 0\), \(1 \leq k \leq n-1\), if \(Z \subset G(n, n-k)\) is a closed set satisfying \(\bigcup_{E \in Z} E \cap S^{n-1} = S^{n-1}\), then there exists a measure \(\mu \in \mathcal{M}_+(G(n, n-k))\) supported in \(Z\), such that \(R^*_{n-k}(d\mu) \geq 1\).

Under this conjecture, we immediately have the following counterpart to Lemma 5.7:

**Lemma 5.8.** Assume the Covering Property Conjecture, and let \(Z \subset G(n, n-k)\) be a closed set satisfying the covering property. Then there exists a measure \(\mu \in \mathcal{M}_+(G(n, n-k))\) supported in \(Z\), such that \(R^*_{n-k}(d\mu) \geq 1\) and \(R^*_k(d\mu^+) \geq 1\).

**Proof.** Apply the Conjecture to the closed sets \(Z \subset G(n, n-k)\) and \(Z^\perp \subset G(n, k)\), and let \(\mu_1 \in \mathcal{M}_+(G(n, n-k))\) and \(\mu_2 \in \mathcal{M}_+(G(n, k))\) be the resulting measures. Then \(\mu_1 + \mu_2^\perp\) is supported in \(Z\) and satisfies the requirements. \(\Box\)

**Remark 5.4.** A very natural way to approach the proof of the Covering Property Conjecture, is to assume that the closed set \(Z\) satisfying \(\bigcup_{E \in Z} E \cap S^{n-1} = S^{n-1}\) is minimal w.r.t. set inclusion (indeed, by Zorn’s lemma it is easy to verify that there exists such a minimal set). The natural candidate for a measure supported on \(Z\) is simply the Hausdorff measure \(H_Z\) on \(Z\), and it remains to show that \(H_Z\) is a finite measure and that \(R^*_{n-k}(dH_Z) \geq \epsilon\) for some \(\epsilon > 0\), using the minimality of \(Z\). In particular, one has to show that the Hausdorff dimension of \(Z\) is \(k\). Although having some progress in this direction, we have not been
able to give a complete proof. We also remark that it is easy to construct a non-bounded measure \( \mu \) supported on \( Z \) for which \( R^*_{n-k}(d\mu) \geq 1 \), simply by using the counting measure on \( Z \), i.e. \( \mu(A) = |\{A \cap Z\}| \) for any Borel set \( A \subset G(n, n-k) \) (where \( |A| \) denotes the cardinality of \( A \)).

As opposed to Theorem 5.5, where \( R^*_{n-k} \) was treated as an operator on \( \mathcal{M}(n, n-k) \), we now go back to the original definition of \( R^*_{n-k} \) as an operator acting on the entire \( \mathcal{M}(G(n, n-k)) \). We summarize this in the following lemma, abbreviating as usual \( G = G(n, n-k) \):

**Lemma 5.9.**

1. \( \mathcal{M}(n, n-k) = \mathcal{M}(G)/\text{Ker}R^*_{n-k} \).
2. \( \mathcal{M}_+(n, n-k) = \{ \mu + \text{Ker}R^*_{n-k} \mid \mu \in \mathcal{M}_+(G) \} \).
3. \( \{ \mu \in \mathcal{M}(G) \mid \langle \mu, f \rangle \geq 0 \forall f \in \overline{R_{n-k}(C(S^{n-1}))}_+ \} = \mathcal{M}_+(G) + \text{Ker}R^*_{n-k} \).

**Proof.** (1) is simply the definition of \( \mathcal{M}(n, n-k) \). (2) follows from (3), since \( \mathcal{M}_+(n, n-k) \) is defined as the cone of non-negative linear functionals on \( \text{Im}R_{n-k} \), and any linear functional on the subspace may be extended to the entire space, hence to \( \mu \in \mathcal{M}(G) \). (3) was already implicitly used in the proof of Lemma 5.2, but we repeat the argument once more. The right-hand set is clearly a subset of the left-hand set, since \( \text{Ker}R^*_{n-k} \) is perpendicular to \( \text{Im}R_{n-k} \) by (5.1). Conversely, any \( \mu \) in the left-hand set is a non-negative linear functional on \( \text{Im}R_{n-k} \), and by a version of the Hahn-Banach Theorem (as in the proof of Lemma 5.2), may be extended to a \( \mu' \in \mathcal{M}_+(G) \). Again by (5.1), the difference \( \mu' - \mu \) must lie in \( \text{Ker}R^*_{n-k} \), concluding the proof.

We now state several more formulations, which are shown to be equivalent each to the other. We then show that under the Covering Property Conjecture, a positive answer to the \( B^p \) question would imply these new statements. For a closed set \( Z \subset G(n, n-k) \), we denote by \( \mathcal{M}(Z) \) the set of all measures in \( \mathcal{M}(G(n, n-k)) \) supported in \( Z \).

**Theorem 5.10.** Let \( n \) and \( 1 \leq k \leq n-1 \) be fixed, and let \( Z \subset G(n, n-k) \) denote a closed subset. Then the following are equivalent:

1. There does not exist a non-zero \( f \in \overline{R_{n-k}(C(S^{n-1}))}_+ \) such that \( Z \subset f^{-1}(0) \).
2. \( \overline{R_{n-k}(C(S^{n-1}))}_+ \cap \{ f \in C(G) \mid f|_Z = 0 \} = \{0\} \).
3. \( \mathcal{M}_+(G) + \text{Ker}R^*_{n-k} + \mathcal{M}(Z) = \mathcal{M}(G) \).
4. There exists a measure \( \mu \in \mathcal{M}(G) \) such that \( R^*_{n-k}(d\mu) = 0 \) and \( \mu = \mu_1 + \mu_2 \) where \( \mu_i \in \mathcal{M}(G) \), \( \mu_1 \geq 1 \) and \( \mu_2 \) is supported in \( Z \).

It is clear that (2) is just a convenient reformulation of (1). We will show that (2) \( \iff \) (3) and (3) \( \iff \) (4).
Proof of (2) \iff (3). Again, we use the Hahn-Banach theorem which shows that for cones, \( P_1 = P_2 \) iff \( P_1^* = P_2^* \). The dual cone (in \( M(G) \)) to \( R_{n-k}(C(S^{n-1}))_+ \) is by definition:

\[
\left\{ \mu \in M(G) \mid \langle \mu, f \rangle \geq 0 \forall f \in R_{n-k}(C(S^{n-1}))_+ \right\},
\]

which by Lemma 5.9 is equal to \( M_+(G) + \text{Ker} R_{n-k}^* \). The dual cone to \( C_Z(G) = \{ f \in C(G) \mid f|_Z = 0 \} \) is obviously \( M(Z) \). Indeed, by definition, if \( \mu \in M(G) \) is not supported in \( Z \), there exists a \( f \in C_Z(G) \) such that \( \langle \mu, f \rangle \neq 0 \) (since \( Z \) is closed). Since also \( -f \in C_Z(G) \), either \( \langle \mu, f \rangle \) or \( \langle \mu, -f \rangle \) is negative, and therefore \( \mu \) cannot be in the dual cone to \( C_Z(G) \). The dual cone to \( \{0\} \) is of course \( M(G) \). Using \( (P_1 \cap P_2)^* = P_1^* + P_2^* \), this concludes the proof. \( \square \)

Proof of (3) \( \Rightarrow \) (4). Apply (3) with the measure \(-1 \in M(G)\) on the right hand side. Then there exist measures \( \nu_1 \in M_+(G) \), \( \nu_2 \in \text{Ker} R_{n-k}^* \) and \( \nu_3 \in M(Z) \), such that \( \nu_1 + \nu_2 + \nu_3 = -1 \). Denoting \( \mu = -\nu_2 \), \( \mu_1 = \nu_1 + 1 \) and \( \mu_2 = \nu_3 \), (4) follows immediately. \( \square \)

Proof of (4) \( \Rightarrow \) (3). \( C(G) \) is dense in \( M(G) \) in the w*-topology, so it is enough to show that (4) implies \( C(G) \subset M_+(G) + \text{Ker} R_{n-k}^* + M(Z) \), as the cones on the right hand side are closed in this topology. Let \( g \in C(G) \), so there exists a constant \( C \geq 0 \) such that \( g + C \geq 0 \), and hence \( g + C + \text{Ker} R_{n-k}^* \in M_+(n,n-k) \). By Lemma 5.9, this means that \( g + C \in M_+(G) + \text{Ker} R_{n-k}^* \), and we see that it is enough to show that the measure \(-C\) is in \( M_+(G) + \text{Ker} R_{n-k}^* + M(Z) \). Since all of the involved sets are cones, it is enough to show the claim for the measure \(-1\). But this follows from formulation (4) in the same manner is in the previous proof. Indeed, let \( \mu = \mu_1 + \mu_2 \) as assured by (4), where \( \mu \in \text{Ker} R_{n-k}^* \), \( \mu_1 - 1 \in M_+(G) \) and \( \mu_2 \in M(Z) \). Then \(-1 = (\mu_1 - 1) - \mu + \mu_2 \in M_+(G) + \text{Ker} R_{n-k}^* + M(Z) \). This concludes the proof. \( \square \)

Comparing formulations (6) in Theorem 5.5 and (1) in Theorem 5.10 for a set \( Z \) satisfying the covering property, and using Lemmas 5.7 and 5.8, the following should now be clear:

**Proposition 5.11.** Let \( n \) and \( 1 \leq k \leq n-1 \) be fixed. Assuming the Covering Property Conjecture, if any of the formulations in Theorem 5.5 hold, then so do any of the formulations in Theorem 5.10 for any closed \( Z \subset G(n,n-k) \) satisfying the covering property.

**Proof.** The statement follows immediately from the remark before the Proposition, taking into account Remark 5.3 and Lemma 5.6. \( \square \)

6. Appendix

In the Appendix, we formulate and prove Proposition 6.1, which is an extended version of the statement from the Introduction and of Corollary 3.2. We have left the proof of Proposition 6.1 for the Appendix, since the technique involved differs from those used in the rest of this note. Although the proposition is of elementary nature and fairly simple to prove, we have not been able to find a reference to it in the literature, so we give a self contained proof here. A similar formulation of the case \( k_1, \ldots, k_r = 1 \) was given by Blaschke and Petkantschin (see [30],[26] for an easy derivation), and used by Grinberg and Zhang in [16] to deduce that \( BP_1^n \subset BP_l^n \) for all \( 1 \leq l \leq n-1 \).

We assume some elementary knowledge of exterior products of differential forms on homogeneous spaces. A rigorous derivation may be found in [30], but we recommend the intuitive exposition in [26, Sections 2,3]. We will also use the notations from Section 3.
We will use the following terminology. For a set of $m$ vectors $\tilde{v} = \{v_1, \ldots, v_m\}$ in a Euclidean space $V$, denote by $Vol_m(\tilde{v}) = det(\{(v_i, v_j)\}_{i,j=1}^{m})^{1/2}$, which is exactly the $m$-dimensional volume of the parallelepiped spanned by $\tilde{v}$. If $m = \sum_{i=1}^{r} k_i$, let $U_i$ be a $k_i$ dimensional subspace of $V$. Choose an arbitrary basis $\tilde{u}^i = \{u^i_1, \ldots, u^i_{k_i}\}$ of $U_i$ such that $Vol_{k_i}(\tilde{u}^i) = 1$, and let $\tilde{u} = \cup_{i=1}^{r} \tilde{u}^i$. Then the $m$-dimensional volume of the parallelepiped spanned by unit volume elements of $U_1, \ldots, U_r$ is defined as $Vol_m(\tilde{u})$. It is easy to verify that this definition indeed does not depend on the basis $\tilde{u}^i$ chosen for $U_i$, as long as $Vol_{k_i}(\tilde{u}^i) = 1$ (this will also be clear from the proof of Proposition 6.1).

**Proposition 6.1.** Let $n > 1$ be fixed, let $d$ be an integer between 0 and $n - 1$, and let $D \in G(n, d)$. For $i = 1, \ldots, r$, let $k_i \geq 1$ denote integers whose sum $l \leq n - d$. For $a = 1, \ldots, n - d$ denote by $G^a = G(n, n - a)$, and by $\mu^a_D$ the Haar probability measure on $G^a_D$. For $F \in G^a_D$ and $a = 1, \ldots, r - 1$, denote by $\mu^a_F$ the Haar probability measure on $G^a_F$. Denote by $\bar{F} = (E_1, \ldots, E_r)$ an ordered set with $E_i \in G^{d_i}$. Then for any continuous function $f(\bar{E}) = f(E_1, \ldots, E_r)$ on $G^{k_1} \times \cdots \times G^{k_r}$:

\[
\int_{E_1 \in G^{k_1}_D} \cdots \int_{E_r \in G^{k_r}_D} f(\bar{E})d\mu^{k_1}_D(E_1) \cdots d\mu^{k_r}_D(E_r) = \int_{F \in G^{a}_D} \int_{E_1 \in G^{d_1}_F} \cdots \int_{E_r \in G^{d_r}_F} f(\bar{E})\Delta(\bar{E})d\mu^{k_1}_F(E_1) \cdots d\mu^{k_r}_F(E_r)d\mu^{l}_D(F),
\]

where $\Delta(\bar{E}) = C_n,_{\{k_i\},l,d}(\bar{E})^{n-d-l}$, $C_n,_{\{k_i\},l,d}$ is a constant depending only on $n, \{k_i\}, l, d,$ and $\Omega(\bar{E})$ denotes the volume of the $l$-dimensional parallelepiped spanned by unit volume elements of $E_1^{+}, \ldots, E_r^{+}$.

**Remark 6.1.** One way to compute the constant $C_n,_{\{k_i\},l,d}$ is to use the function $f = 1$ in Proposition 6.1. Perhaps a better way is to follow the proof, which gives:

\[
C_n,_{\{k_i\},l,d} = \frac{|G(n - d, n - d - l)|\prod_{i=1}^{r} |G(l, l - k_i)|}{\prod_{i=1}^{r} |G(n - d, n - d - k_i)|},
\]

where $|G(a, b)|$ denotes the volume of the Grassmann Manifold $G(a, b)$, and is given by ([26]):

\[
|G(a, b)| = \frac{|S^{a-1}| \cdots |S^{a-b}|}{|S^{b-1}| \cdots |S^{0}|},
\]

where $|S^m|$ denotes the volume of the Euclidean unit sphere $S^m$ of dimension $m$ (and $|S^{0}| = 2$).

**Proof of Proposition 6.1.** We will show that the densities $d\mu^{k_1}_D(E_1) \cdots d\mu^{k_r}_D(E_r)$ and $\Delta(\bar{E})d\mu^{k_1}_F(E_1) \cdots d\mu^{k_r}_F(E_r)d\mu^{l}_D(F)$ with $F = \cap_{a=1}^{r} E_a$ coincide on a set of measure 1 w.r.t. both measures. It is easy to verify that the set consisting of all $(E_1, \ldots, E_r)$ such that $dim(\cap_{a=1}^{r} E_a) = n - l$ satisfies this requirement, and therefore $F$ above is in $G(n, n - l)$, hence the second measure is well defined. Indeed, this set is exactly complementary to the set of all $(E_1, \ldots, E_r)$ such that $\Omega(\bar{E}) = 0$, which defines a lower dimensional analytic submanifold of $G^{k_1} \times \cdots \times G^{k_r}$, hence having measure 0 w.r.t. the first (Haar) measure.
If $J \in G(a,c)$, it is well-known ([26]) that the volume element of $G_J(a,b)$ for $b > c$ at $H \in G_J(a,b)$ is given by:

\begin{equation}
(6.2) \quad dG_J(a,b) = \bigwedge_{i=c+1}^{b} \bigwedge_{j=b+1}^{a} w_{i,j},
\end{equation}

where $w_{i,j} = \langle e_i, de_j \rangle$, and \{e_1, \ldots, e_a\} is any orthonormal basis of $\mathbb{R}^a$ such that $J = \text{span} \{e_1, \ldots, e_c\}$ and $H = \text{span} \{e_1, \ldots, e_b\}$. Indeed, it is easy to verify that this formula does not depend on the given orthonormal basis satisfying these conditions, by changing basis and applying a change of variables formula. With this normalization, the total volume of $G_J(a,b)$ is $|G(a-c, b-c)|$, as defined in (6.1) ([26]). Since $d_1 \wedge d_2 = -d_2 \wedge d_1$, the volume element is signed, corresponding to the assumed orientation of the element. However, we will henceforth ignore the orientation and implicitly take the absolute value in all exterior products, except where it is mentioned otherwise. Note also that the skew-symmetry implies $d_1 \wedge d = 0$.

Let \{f_1, \ldots, f_d\} be an orthonormal basis of $D$, and let \{f_1, \ldots, f_{n-1}\} be a completion to an orthonormal basis of $F$. For $a = 1, \ldots, r$ let \{e_{n-l+1}^{a}, \ldots, e_{n-k_a+1}^{a}\} be an orthonormal basis of $F^1 \cap E_a$, and let \{e_{n-k_a+1}^{a}, \ldots, e_{n}^{a}\} be an orthonormal basis of $E_a^1$. For every $a$ we define $e_i^a = f_i$ for $i = 1, \ldots, n-l$. Then:

\begin{equation}
\mu_{D}^k (E_1) \cdots \mu_{D}^k (E_r) = C_{n, \{k_i\}, l, d}^1 \bigwedge_{a=1}^{r} \bigwedge_{i=d+1}^{a} \bigwedge_{j=n-k_a+1}^{n} w_{i,j}^a,
\end{equation}

where $w_{i,j}^a = \langle e_i^a, de_j^a \rangle$ and $C_{n, \{k_i\}, l, d}^1 = (\Pi_{i=1}^{r} |G(n-d, n-d-k_i)|)^{-1}$ accounts for the fact that the measure on the left is normalized to have total mass 1. Notice that by (6.2):

\begin{equation}
\bigwedge_{a=1}^{r} \bigwedge_{i=n-l+1}^{a} \bigwedge_{j=n-k_a+1}^{n} w_{i,j}^a = C_{n, \{k_i\}, l, d}^2 \Delta(E) \mu_{F}^l (F),
\end{equation}

where $C_{n, \{k_i\}, l, d}^2 = \Pi_{i=1}^{r} |G(l, l-k_i)|$. It remains to show that:

\begin{equation}
(6.3) \quad \bigwedge_{a=1}^{r} \bigwedge_{i=d+1}^{a} \bigwedge_{j=n-k_a+1}^{n} w_{i,j}^a = C_{n, \{k_i\}, l, d}^2 \Delta(E) \mu_{F}^l (F).
\end{equation}

Now let \{g_{n-l+1}, \ldots, g_n\} denote an orthonormal basis of $F^1$, and denote $\lambda_{j,v}^a = \langle e_j^a, g_v \rangle$ for $j, v = n - l + 1, \ldots, n$. Hence $e_j^a = \sum_{v=n}^{n-l} \lambda_{j,v}^a g_v$ and $de_j^a = \sum_{v=n}^{n-l}(d \lambda_{j,v}^a g_v + \lambda_{j,v}^a d g_v)$. Denoting $w_{j,v} = \langle f_j, dg_v \rangle$, we see that since $\langle f_i, g_v \rangle = 0$, then for $i = 1, \ldots, n-l$ and $j = n - l + 1, \ldots, n$:

\begin{equation}
(6.4) \quad w_{i,j}^a = \sum_{v=n-l+1}^{n} \lambda_{j,v}^a w_{i,v}.
\end{equation}

As evident from (6.3), we will be interested in the values of $\lambda_{j,v}^a$ only in the range $j = n - k_a + 1, \ldots, n$. We therefore rearrange these values by defining a bijection:

\begin{equation}
u : \cup_{a=1}^{n} \{(a, n-k_a+1), \ldots, (a, n)\} \rightarrow \{1, \ldots, l\},
\end{equation}
and denote $\Lambda_{u(a,j),v} = \lambda_{j,v}^a$. Plugging (6.4) into (6.3), we have:

$$\begin{align*}
\bigwedge_{a=1}^r \bigwedge_{i=1}^{n-l} \bigwedge_{j=n-k_a+1}^n w_{i,j}^a &= \bigwedge_{i=d+1}^{n-l} \bigwedge_{j=n-k_a+1}^n \sum_{v=n-l+1}^n \lambda_{j,v}^a w_{i,v} = \\
\bigwedge_{i=d+1}^{n-l} \bigwedge_{j=n-k_a+1}^n \sum_{v=n-l+1}^n \Lambda_{u,v} w_{i,v} &= \sum_{i=d+1}^{n-l} \det(\Lambda) w_{i,n-l+1} \wedge \ldots \wedge w_{i,n}.
\end{align*}$$

The last transition is standard and is explained by the skew-symmetry of the exterior product: all terms for which $w_{i,v_1} \wedge \ldots \wedge w_{i,v_l}$ contains a recurring $v_i = v_j$ are 0, and we are only left with the case $v_i = \pi(i)$, where $\pi$ is a permutation of $\{n - l + 1, \ldots, n\}$; these terms are equal to $(-1)^{\text{sign}(\pi)} w_{i,n-l+1} \wedge \ldots \wedge w_{i,n}$, producing the determinant of $\Lambda$. Continuing, since $\Lambda$ does not depend on $i$ and using (6.2), we see that:

$$\begin{align*}
\bigwedge_{a=1}^r \bigwedge_{i=1}^{n-l} \bigwedge_{j=n-k_a+1}^n w_{i,j}^a &= \det(\Lambda)^{n-l-d} \bigwedge_{i=d+1}^{n-l} \bigwedge_{j=n-k_a+1}^n w_{i,j} = \det(\Lambda)^{n-l-d} C_{n,l,d}^3 \mu_D(F),
\end{align*}$$

where $C_{n,l,d}^3 = |G(n-d,n-d-l)|$. To deduce (6.3), it remains to show that $\det(\Lambda) = \Omega(E)$.

Recall that $\lambda_{j,v}^a = \left\langle e_{j}^a, \phi_v \right\rangle$, and in the range $j = n - k_a + 1, \ldots, n$, these are exactly the coefficients of the orthonormal bases $\vec{e}^a = \{\vec{e}^{n-k_a+1}_a, \ldots, \vec{e}_a^n\}$ of $E_a^\perp$ w.r.t. the orthonormal basis $\vec{g} = \{g_{n-l+1}, \ldots, g_n\}$ of $F^\perp$. Using the orthogonality of $\vec{g}$, it is immediate that $(\Lambda \Lambda^t)_{u(a_1;j_1),u(a_2;j_2)} = \left\langle e_{j_1}^{a_1}, e_{j_2}^{a_2} \right\rangle$, and therefore $\det(\Lambda) = \text{Vol}_{F^\perp}(\vec{e})$ for $\vec{e} = \{\vec{e}_1, \ldots, \vec{e}_n\}$, which is exactly the definition of $\Omega(E)$. Incidentally, this also shows that $\text{Vol}_{F^\perp}(\vec{e})$ is invariant to taking an arbitrary (not necessarily orthonormal) basis $\vec{e}^a$ of $E_a^\perp$ with $\text{Vol}_{E^\perp}(\vec{e}^a) = 1$, since this is easily checked for $\det(\Lambda)$.

\[ \square \]

REFERENCES


11. A. A. Giannopoulos, *A note on a problem of h. busemann and c.m. petty concerning sections of symmetric convex bodies*, Mathematika 37 (1990), 239–244.

E-mail address: emanuel.milman@weizmann.ac.il

DEPARTMENT OF MATHEMATICS, THE WEIZMANN INSTITUTE OF SCIENCE, REHOVOT 76100, ISRAEL.
CHAPTER 2

GENERALIZED INTERSECTION BODIES ARE NOT EQUIVALENT

EMANUEL MILMAN

To appear in Advances in Mathematics

Abstract. In [17], A. Koldobsky asked whether two types of generalizations of the notion of an intersection-body, are in fact equivalent. The structures of these two types of generalized intersection-bodies have been studied by the author in [22], providing substantial evidence for a positive answer to this question. The purpose of this note is to construct a counter-example, which provides a surprising negative answer to this question in a strong sense. This implies the existence of non-trivial non-negative functions in the range of the spherical Radon transform, and the existence of non-trivial spaces which embed in $L^p$ for certain negative values of $p$.

1. Introduction

Let $\text{Vol}(L)$ denote the Lebesgue measure of a set $L \subset \mathbb{R}^n$ in its affine hull, and let $G(n,k)$ denote the Grassmann manifold of $k$ dimensional subspaces of $\mathbb{R}^n$. Let $D_n$ denote the Euclidean unit ball, and $S^{n-1}$ the Euclidean sphere. All of the bodies considered in this note will be assumed to be centrally-symmetric star-bodies (even if the central-symmetry assumption is omitted). A centrally-symmetric star-body $K$ is a compact set with non-empty interior such that $K = -K$, $tK \subset K$ for all $t \in [0,1]$, and such that its radial function $\rho_K(\theta) = \max \{ r \geq 0 \mid r\theta \in K \}$ for $\theta \in S^{n-1}$ is an even continuous function on $S^{n-1}$.

This note concerns two generalizations of the notion of an intersection body, first introduced by E. Lutwak in [20] (see also [21]).

Definition. A star-body $K$ is said to be an intersection body of a star-body $L$, if $\rho_K(\theta) = \text{Vol}(L \cap \theta^\perp)$ for every $\theta \in S^{n-1}$, where $\theta^\perp$ is the hyperplane perpendicular to $\theta$. $K$ is said to be an intersection body, if it is the limit in the radial metric $d_r$ of intersection bodies $\{K_i\}$ of star-bodies $\{L_i\}$, where $d_r(K_1,K_2) = \sup_{\theta \in S^{n-1}} |\rho_{K_1}(\theta) - \rho_{K_2}(\theta)|$.

Let $R : C(S^{n-1}) \to C(S^{n-1})$ denote the Spherical Radon Transform, defined by:

$$R(f)(\theta) = \int_{S^{n-1} \cap \theta^\perp} f(\xi) d\sigma_\theta(\xi)$$

for $f \in C(S^{n-1})$, where $\sigma_\theta$ denotes the Haar probability measure on $S^{n-1} \cap \theta^\perp$. Let $R^*$ denote the dual transform (as in (1.2) below). We will use the following characterization (see [21], [5]) as an equivalent definition:

Supported in part by BSF and ISF.
Equivalent Definition. A star-body $K$ is an intersection body iff $\rho_K = R^*(d\mu)$, where $\mu$ is a non-negative Borel measure on $S^{n-1}$.

The notion of an intersection body has been shown to be fundamentally connected to the Busemann-Petty Problem (first posed in [4]), which asks whether two centrally-symmetric convex bodies $K$ and $L$ in $\mathbb{R}^n$ satisfying:

$$\text{Vol}(K \cap H) \leq \text{Vol}(L \cap H) \quad \forall H \in G(n, n-1)$$

necessarily satisfy $\text{Vol}(K) \leq \text{Vol}(L)$. It was shown in [21], [5] that the answer is equivalent to whether all centrally-symmetric convex bodies in $\mathbb{R}^n$ are intersection bodies, and in a series of results ([19], [1], [2], [10], [24], [5], [6], [14], [30], [7]) that this is true for $n \leq 4$, but false for $n \geq 5$.

In [29], G. Zhang considered a generalization of the Busemann-Petty problem, the so-called generalized $k$-codimensional problem, asking whether two centrally-symmetric convex bodies $K$ and $L$ in $\mathbb{R}^n$ satisfying:

$$\text{Vol}(K \cap H) \leq \text{Vol}(L \cap H) \quad \forall H \in G(n, n-k)$$

for some fixed integer $k$ between 1 and $n-1$, necessarily satisfy $\text{Vol}(K) \leq \text{Vol}(L)$. Zhang showed that this generalized problem is also naturally associated to a class of generalized intersection-bodies, which will be referred to as $k$-Busemann-Petty bodies, and that the generalized $k$-codimensional problem is equivalent to whether all centrally-symmetric convex bodies in $\mathbb{R}^n$ are $k$-Busemann-Petty bodies. It was shown in [3] (see also [25]), and later in [17], that the answer is negative for $k < n-3$, but the cases $k = n-3$ and $k = n-2$ remain open (the case $k = n-1$ is obviously true).

Several partial answers to these unresolved cases are known. It was shown in [29] (see also [25]) that when $K$ is a centrally-symmetric convex body of revolution then the answer is positive for the pair $K, L$ with $k = n-2, n-3$ and any star-body $L$. When $k = n-2$, it was shown in [3] that the answer is positive if $L$ is a Euclidean ball and $K$ is convex and sufficiently close to $L$. This was extended in [23], where it was shown that this is again true for $k = n-2$ and $k = n-3$, when $L$ is an arbitrary star-body and $K$ is sufficiently close to a Euclidean ball (but to an extent depending on its curvature). Several other generalizations of the Busemann-Petty problem were treated in [25], [31], [27], [28].

Before defining the class of $k$-Busemann-Petty bodies we shall need to introduce the $m$-dimensional Spherical Radon Transform, acting on spaces of continuous functions as follows:

$$R_m : C(S^{n-1}) \longrightarrow C(G(n, m))$$

$$R_m(f)(E) = \int_{S^{n-1} \cap E} f(\theta)d\sigma_E(\theta),$$

where $\sigma_E$ is the Haar probability measure on $S^{n-1} \cap E$. It is well known (e.g. [13]) that as an operator on even continuous functions, $R_m$ is injective. The dual transform is defined on spaces of signed Borel measures $\mathcal{M}$ by:

$$R_m^* : \mathcal{M}(G(n, m)) \longrightarrow \mathcal{M}(S^{n-1})$$

$$\int_{S^{n-1}} f R_m^*(d\mu) = \int_{G(n, m)} R_m(f)d\mu \quad \forall f \in C(S^{n-1}),$$
and for a measure $\mu$ with continuous density $g$, the transform may be explicitly written in terms of $g$ (see [29]):

$$R^*_m,g(\theta) = \int_{\theta \in E \subset G(n,m)} g(E)d\nu_{m,\theta}(E),$$

where $\nu_{m,\theta}$ is the Haar probability measure on the homogeneous space $\{E \in G(n, m) \mid \theta \in E\}$.

**Definition.** A star-body $K$ in $\mathbb{R}^n$ is called a $k$-Busemann-Petty body if $\rho^K_k = R^*_{n-k}(d\mu)$ as measures in $\mathcal{M}(S^{n-1})$, where $\mu$ is a non-negative Borel measure on $G(n, n-k)$. This class of bodies is denoted by $\mathcal{BP}^k_n$.

Choosing $k = 1$, for which $G(n, n-1)$ is isometric to $S^{n-1}/Z_2$ by mapping $H$ to $S^{n-1}\cap H^\perp$, and noticing that $R$ is equivalent to $R_{n-1}$ under this map, we see that $\mathcal{BP}^1_n$ is exactly the class of intersection bodies.

In [17], a second generalization of the notion of an intersection body was introduced by A. Koldobsky, who studied a different analytic generalization of the Busemann-Petty problem.

**Definition.** A centrally-symmetric star-body $K$ is said to be a $k$-intersection body of a star-body $L$, if $\text{Vol} (K \cap H^\perp) = \text{Vol} (L \cap H)$ for every $H \in G(n, n-k)$. $K$ is said to be a $k$-intersection body, if it is the limit in the radial metric $d_r$ of $k$-intersection bodies $\{K_i\}$ of star-bodies $\{L_i\}$. We shall denote the class of such bodies by $\mathcal{I}_k^n$.

Again, choosing $k = 1$, we see that $\mathcal{I}_1^n$ is exactly the class of intersection bodies.

In [17], Koldobsky considered the relationship between these two types of generalizations, $\mathcal{BP}^k_n$ and $\mathcal{I}_k^n$, and proved that $\mathcal{BP}^k_n \subset \mathcal{I}_k^n$ (see also [22]). Koldobsky also asked whether the opposite inclusion is equally true for all $k$ between 2 and $n-2$ (for 1 and $n-1$ this is true):

**Question ([17]):** Is it true that $\mathcal{BP}^k_n = \mathcal{I}_k^n$ for $n \geq 4$ and $2 \leq k \leq n - 2$ ?

If this were true, as remarked by Koldobsky, a positive answer to the generalized $k$-codimensional Busemann-Petty problem for $k \geq n - 3$ would follow, since for those values of $k$ any centrally-symmetric convex body in $\mathbb{R}^n$ is known to be a $k$-intersection body ([15],[16], [17]).

In [22], it was shown that these two classes $\mathcal{BP}^k_n$ and $\mathcal{I}_k^n$ share many identical structural properties, suggesting that it is indeed reasonable to believe that $\mathcal{BP}^k_n = \mathcal{I}_k^n$. Using techniques from Integral Geometry for the class $\mathcal{BP}^k_n$ and Fourier transform of distributions techniques for the class $\mathcal{I}_k^n$, the following structure Theorem was established (see [22] for an account of particular cases which were known before). We define the $k$-radial sum of two star-bodies $L_1, L_2$ as the star-body $L$ satisfying $\rho^k_L = \rho^k_{L_1} + \rho^k_{L_2}$.

**Structure Theorem ([22])** Let $C = I$ or $C = \mathcal{BP}$ and $k, l = 1, \ldots, n-1$. Then:

1. $C^0_k$ is closed under full-rank linear transformations, $k$-radial sums and taking limit in the radial metric.
2. $C^1_k$ is the class of intersection-bodies in $\mathbb{R}^n$, and $C^0_{n-1}$ is the class of all symmetric star-bodies in $\mathbb{R}^n$.
3. Let $K_1 \in C^0_{k_1}$, $K_2 \in C^0_{k_2}$ and $l = k_1 + k_2 \leq n - 1$. Then the star-body $L$ defined by $\rho^l_L = \rho^{k_1}_{K_1} + \rho^{k_2}_{K_2}$ satisfies $L \in C^0_l$. As corollaries:
   a) $C^0_{k_1} \cap C^0_{k_2} \subset C^0_{k_1 + k_2}$ if $k_1 + k_2 \leq n - 1$. 

2. GENERALIZED INTERSECTION BODIES ARE NOT EQUIVALENT 52
(b) $C^n_k \subset C^n_l$ if $k$ divides $l$.
(c) If $K \in C^n_k$ then the star-body $L$ defined by $\rho_L = \rho_K^{kl}$ satisfies $L \in C^n_l$ for $l \geq k$.
(d) If $K \in C^n_k$ then any $m$-dimensional central section $L$ of $K$ (for $m > k$) satisfies $L \in C^n_k$.

Despite this and other evidence from [22] for a positive answer to Koldobsky’s question, we give the following negative answer. Let $O(n)$ denote the orthogonal group on $\mathbb{R}^n$. Recall that a star-body $K$ is called a body of revolution if its radial function $\rho_K \in C(S^{n-1})$ is invariant under the natural action of $O(n-1)$ identified as some subgroup of $O(n)$.

**Theorem 1.1.** Let $n \geq 4$ and $2 \leq k \leq n-2$. Then there exists an infinitely smooth centrally-symmetric body of revolution $K$ such that $K \in I^n_k$, but $K \notin BP^n_k$.

Note that Theorem 1.1 does not imply a negative answer to the unresolved cases $k = n-2, n-3$ (for $n \geq 5$) of the generalized Busemann-Petty problem, which pertains to convex bodies. Indeed, the $K$ we construct cannot be a convex body in those ranges of $k$, since as already mentioned, convex bodies of revolution are known ([29], see also [25]) to belong to $BP^n_{n-2}$ and $BP^n_{n-3}$. Theorem 1.1 does however imply that if one wishes to prove a positive answer to these unresolved cases by means of comparing $k$-intersection bodies to $k$-Busemann-Petty bodies, it is essential to restrict one’s attention to convex bodies.

Let $I : C(G(n, k)) \to C(G(n, n-k))$ denote the operator defined by $I(f)(E) = f(E^\perp)$ for all $E \in G(n, n-k)$. Let $R_{n-k}(C(S^{n-1})) = \text{Im } R_{n-k}$ denote the range of $R_{n-k}$. As explained in Section 2, Theorem 1.1 can be equivalently reformulated as follows:

**Theorem 1.2.** Let $n \geq 4$ and $2 \leq k \leq n-2$. Then there exists an infinitely smooth function $g \in C(G(n, n-k))$ such that $R^*_{n-k}(g) \geq 1$ and $(I \circ R_k)^* (g) \geq 1$ as functions in $C(S^{n-1})$, but $g$ is not non-negative as a functional on $R_{n-k}(C(S^{n-1}))$. In other words, there exists a non-negative $h \in R_{n-k}(C(S^{n-1}))$ such that $\int_{G(n, n-k)} g(E) h(E) d\eta_{n,n-k}(E) < 0$, where $\eta_{n,n-k}$ is the Haar measure on $G(n, n-k)$. Moreover, both $g$ and $h$ can be chosen to be invariant under the action of $O(n-1)$.

In [22], several equivalent formulations to Koldobsky’s question were obtained using cone-duality and the Hahn-Banach Theorem. Let $C_+(S^{n-1})$ denote the cone of non-negative continuous functions on the sphere, and let $R_{n-k}(C(S^{n-1}))_+$ denote the cone of non-negative functions in the image of $R_{n-k}$. Let $\bar{A}$ denote the closure of a set $A$ in the corresponding normed space. Note that by the results from [22], $\text{Im } I \circ R_k = \text{Im } R_{n-k}$, and hence:

$$\bar{R}_{n-k}(C(S^{n-1}))_+ \supset R_{n-k}(C_+(S^{n-1}))_+ \cup I \circ R_k(C_+(S^{n-1}))$$

As formally verified in [22], the dual formulation to Theorem 1.2 then reads:

**Theorem 1.3.** Let $n \geq 4$ and $2 \leq k \leq n-2$. Then:

$$R_{n-k}(C(S^{n-1}))_+ \backslash \bar{R}_{n-k}(C_+(S^{n-1}))_+ \cup I \circ R_k(C_+(S^{n-1})) \neq \emptyset$$

In other words, there exists an (infinitely smooth) function $f \in R_{n-k}(C(S^{n-1}))_+$ which can not be approximated (in $C(G(n, n-k))$) by functions of the form $R_{n-k}(g) + I \circ R_k(h)$ with $g, h \in C_+(S^{n-1})$. 
2. GENERALIZED INTERSECTION BODIES ARE NOT EQUIVALENT 54

Other equivalent formulations using the language of Fourier transforms of homogeneous distributions are given in section 5. We comment here that one such formulation pertains to embeddings in $L_p$ for negative values of $p$. The definition of embedding into such a space (for $-n < p < 0$) was given by Koldobsky in [17] by means of analytic continuation of the usual definition for $p > 0$. It is known (see Section 5) that for $p \geq -1$ ($p \neq 0$) and for $-n < p \leq -n + 1$, any star-body $K$ such that $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in $L_p$ can be generated in a “trivial” manner, by starting with the Euclidean ball $D_n$, applying full-rank linear transformations, $(-p)$-radial sums and taking the limit in the radial metric. Our results imply that $p = -1$ and $p = -n + 1$ are critical values for this property, and that this is no longer true for $p = -k$, $2 \leq k \leq n - 2$. In other words:

**Theorem 1.4.** There exist “non-trivial” $n$-dimensional spaces which embed in $L_{-k}$ for $2 \leq k \leq n - 2$.

The rest of this note is organized as follows. In Section 2, we provide some additional background which is required to see why Theorem 1.2 implies Theorem 1.1 and Theorem 1.3. In Section 3, we develop several formulas for the Spherical Radon Transform and its dual for functions of revolution, i.e. functions invariant under the action of $O(n-1)$. In Section 4, we use these formulas to prove Theorem 1.2, thereby constructing the desired counter-example to Koldobsky’s question. In Section 5, we give several additional equivalent formulations to Theorem 1.1 using the language of Fourier transforms of homogeneous distributions.

**Acknowledgments.** I would like to sincerely thank my supervisor Gideon Schechtman
for his guidance. I would also like to thank Alexander Koldobsky for encouraging me to
think about bodies of revolution. Part of this work was done while the author enjoyed the
hospitality of the Mathematisches Forschungsinstitut Oberwolfach.

2. ADDITIONAL BACKGROUND

In this section, we summarize the relevant results needed for this note. We also explain why Theorem 1.1 and 1.3 follow from Theorem 1.2. We refer to [22] for more details.

For a star-body $K$ (not necessarily convex), we define its Minkowski functional as $\|x\|_K = \min \{t \geq 0 \mid x \in tK\}$. When $K$ is a centrally-symmetric convex body, this of course coincides with the natural norm associated with it. Obviously $\rho_K(\theta) = \|\theta\|_K^1$ for $\theta \in S^{n-1}$.

It was shown by Koldobsky in [17] that for a star-body $K$ in $\mathbb{R}^n$, $K \in I^n_k$ iff $\|\cdot\|^{-k}_K$ is a positive definite distribution on $\mathbb{R}^n$, meaning that its Fourier transform (as a distribution) $(\|\cdot\|^{-k}_K)$ is a non-negative Borel measure on $\mathbb{R}^n$. We refer the reader to Section 5 for more on Fourier transforms of homogeneous distributions, as this will not be of essence in the ensuing discussion. To translate this result to the language of Radon transforms, it was shown in [22, Corollary 4.2] that for an infinitely smooth star-body $K$ and a (signed) Borel measure $\mu \in \mathcal{M}(G(n, n - k))$:

$$\|\cdot\|^{-k}_K = R^*_n \mu(d\mu) \text{ iff } (\|\cdot\|^{-k}_K) = c(n, k)(I \circ R)^*_k(d\mu),$$

where $c(n, k)$ is some positive constant and the equalities above are interpreted as equalities between measures on $S^{n-1}$. Hence, it follows ([22, Lemma 5.3]) that for an infinitely smooth star-body $K$ in $\mathbb{R}^n$, $K \in I^n_k$ iff there exists a (possibly signed) Borel measure $\mu \in \mathcal{M}(G(n, n - k))$, such that as measures $\rho^k_K = R^*_n \mu(d\mu) \geq 0$ and $(I \circ R^*_k)(d\mu) \geq 0$. 
This should be compared with the definition of $k$-Busemann-Petty bodies: $K \in \mathcal{BP}_k^n$ iff 
$$
\rho_K^k = R_{n-k}(d\mu) \text{ as measures on } S^{n-1}\text{ for a non-negative Borel measure } \mu \in \mathcal{M}(G(n, n-k)).
$$
Since for such a measure, $(I \circ R_k)^* (d\mu) \geq 0$, it follows that every infinitely smooth $k$-
Busemann-Petty body is also a $k$-intersection body, and this easily implies (see [22, Corollary
4.4]) that $\mathcal{BP}_k^n \subset T_k^n$ in general, as first showed by Koldobsky in [17].

$R_{n-k}$ is known (e.g. [13]) to be injective on the space of even measures in $\mathcal{M}(S^{n-1})$, so by
duality $R_{n-k}^*$ is onto a dense subset of even measures in $\mathcal{M}(S^{n-1})$, which is known to include
even measures with infinitely smooth densities. However, it is important to note that for
$2 \leq k \leq n-2$, the image of $R_{n-k}$ is not dense in $C(G(n, n-k))$, and equivalently, $R_{n-k}^*$ has
a non-trivial kernel. The above implies that for any infinitely smooth star-body $K$, we can
find a measure $\mu$ such that $\rho_K^k = R_{n-k}^*(d\mu)$, but if $2 \leq k \leq n-2$ this measure will not unique.

Nevertheless, as a functional on $R_{n-k}^*(C(S^{n-1}))$, such a measure $\mu$ is determined uniquely.
The conclusion is that if we need to determine whether $K \in \mathcal{BP}_k^n$ given a representation
$\rho_K^k = R_{n-k}^*(d\mu)$ for some measure $\mu \in \mathcal{M}(G(n, n-k))$, a necessary and sufficient condition is
that $\mu$ is a non-negative functional on $R_{n-k}^*(C(S^{n-1}))$, i.e. \( \int_{G(n, n-k)} R_{n-k}^*(h)(E)d\mu(E) \geq 0 \)
for any $h \in C(S^{n-1})$ such that $R_{n-k}^*(h) \geq 0$. Indeed, any non-negative functional on
$R_{n-k}^*(C(S^{n-1}))$ can be extended to a non-negative functional on $C(G(n, n-k))$ by a version of
the Hahn-Banach Theorem (see the remarks before [22, Lemma 5.2] for more details).

The above discussion explains why Theorem 1.1 is an immediate consequence of Theorem
1.2. Given the infinitely smooth function $g$ provided by Theorem 1.2, we define the centrally-
symmetric star-body $K$ given by $\rho_K^k = R_{n-k}^*(g)$. Note that this indeed defines a star-body
since $R_{n-k}^*(g) \geq 0$. In fact, $K$ is an infinitely smooth star-body since it is known (e.g. [8])
that $R_{n-k}^*(g)$ is an infinitely smooth functional on $S^{n-1}$ if $g$ is infinitely smooth; and since
$\rho_K^k = R_{n-k}^*(g) \geq 1$, it follows that $\rho_K$ itself is infinitely smooth. In addition $K \in T_k^n$ since
$(I \circ R_k)^* (g) \geq 0$. But since $g$ is not a non-negative functional on $R_{n-k}^*(C(S^{n-1}))$, it follows
that $K \notin \mathcal{BP}_k^n$.

To explain why Theorem 1.1 is equivalent to Theorem 1.3, we recall another result from
[22]. Denote $\mathcal{M} = M(G(n, n-k))$ for short, and let:

$$
\mathcal{M}(\mathcal{BP}_k^n) = \left\{ \mu \in \mathcal{M}; \mu \text{ is a non-negative functional on } R_{n-k}^*(C(S^{n-1})) \right\},
$$

and:

$$
\mathcal{M}(T_k^n) = \left\{ \mu \in \mathcal{M}; R_{n-k}^*(d\mu) \geq 0 \text{ and } (I \circ R_k)^*(d\mu) \geq 0 \right\}.
$$

It should already be clear from the above discussion that the statement $\mathcal{BP}_k^n = T_k^n$ is equivalent to
the statement $\mathcal{M}(\mathcal{BP}_k^n) = \mathcal{M}(T_k^n)$. By the Hahn-Banach Theorem for convex
cones, it is not hard to see ([22, Theorem 5.6]) that the latter statement is dual to:

$$
R_{n-k}(C(S^{n-1})) = R_{n-k}(C^+(S^{n-1})) + I \circ R_k(C^+(S^{n-1})).
$$

(2.2)

As follows from (2.1), $\text{Ker } R_{n-k}^* = \text{Ker } (I \circ R_k)^*$, and therefore $\text{Im } R_{n-k} = \text{Im } I \circ R_k$.
This explains why the right-hand side of (2.2) is always a subset of the left. Theorem 1.1 shows that it is a proper subset, implying Theorem 1.3. Since this Theorem is attained using a
convex separation argument, we have no constructive way of finding the function $f$
of the Theorem. Albeit, we can always find an infinitely smooth $f$, since the subspace of
infinitely smooth functions in $R_{n-k}(C(S^{n-1}))$ is known to be dense in $R_{n-k}(C(S^{n-1}))$, and
hence in $R_{n-k}(C(S^{n-1}))$. 2. GENERALIZED INTERSECTION BODIES ARE NOT EQUIVALENT 55
3. Radon Transform for Functions of Revolution

Fix $n \geq 3$ and $\xi_0 \in S^{n-1}$. We denote by $O_{\xi_0}(n-1)$ the subgroup of $O(n)$ whose natural action on $S^{n-1}$ leaves $\xi_0$ invariant, and by $C_{\xi_0}(S^{n-1})$ the linear subspace of functions in $C_c(S^{n-1})$ invariant under $O_{\xi_0}(n-1)$. Clearly $O_{\xi_0}(n-1)$ is isometric to $O(n-1)$. We refer to members of $C_{\xi_0}(S^{n-1})$ as spherical functions of revolution. For $\xi_1, \xi_2 \in S^{n-1}$, let $\langle \xi_1, \xi_2 \rangle$ denote the angle in $[0, \pi/2]$ between $\xi_1$ and $\xi_2$, i.e. $\cos \langle \xi_1, \xi_2 \rangle = |\langle \xi_1, \xi_2 \rangle|$. We also denote $\langle \xi_1, 0 \rangle = \pi/2$. Clearly $F \in C_{\xi_0}(S^{n-1})$ iff $F(\xi) = f(\langle \xi, \xi_0 \rangle)$ for $f \in C([0, \pi/2])$. In that case, we denote by $\tilde{f} \in C([0,1])$ the function given by $\tilde{f}(\cos \theta) = f(\theta)$, so $F(\xi) = \tilde{f}(\langle \xi, \xi_0 \rangle)$. We denote the operator $T : C([0,\pi/2]) \to C([0,1])$ defined by $T(f) = \tilde{f}$, for future reference.

It is well known by polar integration (e.g. [26]), that:

$$
\int_{S^{n-1}} F(\xi) d\sigma_n(\xi) = c_n \int_0^{\pi/2} f(\theta) \sin^{n-2}(\theta) d\theta = c_n \int_0^1 \tilde{f}(t)(1-t^2)^{n-3} dt,
$$

where $\sigma_n$ is the Haar probability measure on $S^{n-1}$ and $c_n$ is a constant whose value can be deduced by using $F \equiv f \equiv \tilde{f} \equiv 1$.

For $E \subset G(n,k)$ and $\xi \in S^{n-1}$, denote by $\text{Proj}_E \xi$ the orthogonal projection of $\xi$ onto $E$, and by $\text{Proj}_E^k \xi := \text{Proj}_E \xi / |\text{Proj}_E \xi|$ if $\text{Proj}_E \xi \neq 0$, and $\text{Proj}_E^k \xi := 0$ otherwise. When $E = \text{span}(\xi_1)$ for $\xi_1 \in S^{n-1}$, we may sometimes replace $E$ by $\xi_1$ in $\text{Proj}_E$ and $\text{Proj}_E^k$. Denote by $\langle \xi, E \rangle = \langle \xi, \text{Proj}_E \xi \rangle$ if $\text{Proj}_E \xi \neq 0$ and $\langle \xi, E \rangle = \pi/2$ otherwise.

Since the natural action of $O(n)$ on $C(G(n,k))$ and $C_c(S^{n-1})$ commutes with $R_k$, and since $O_{\xi_0}(n-1)$ acts transitively on all $E \in G(n,k)$ such that $\langle \xi_0, E \rangle$ is fixed, it clearly follows that if $F \in C_{\xi_0}(S^{n-1})$ then $R_k(F)(E)$ only depends on $\langle \xi_0, E \rangle$. Hence, if $F(\xi) = f(\langle \xi, \xi_0 \rangle)$ for $f \in C([0, \pi/2])$, we denote (abusing notation) by $R_k(f) \in C([0, \pi/2])$ the function given by $R_k(f)(\langle \xi_0, E \rangle) = R_k(F)(E)$. Similarly, we define $R_k : C([0,1]) \to C([0,1])$ as $R_k = T \circ R_k \circ T^{-1}$.

The following lemma was essentially stated in [29]. We provide a simple proof for completeness:

**Lemma 3.1.** Let $f \in C([0,\pi/2])$ and $2 \leq k \leq n-1$. Then:

$$
R_k(f)(\phi) = c_k \int_0^{\pi/2} f(\cos^{-1}(\cos \phi \cos \theta)) \sin^{k-2} \theta d\theta,
$$

where the value of $c_k$ is found by using $f \equiv 1$, in which case $R_k(f) \equiv 1$.

**Remark 3.1.** This lemma, together with the subsequent ones, extend to the case $k = 1$, if we properly interpret the (formally) diverging integral as integration with respect to an appropriate delta-measure. Note also that the value $c_k$ is consistent with the one used in (3.1).

**Proof.** Let $F \in C_{\xi_0}(S^{n-1})$ be given by $F(\xi) = f(\langle \xi, \xi_0 \rangle)$. Let $E \subset G(n,k)$ be such that $\langle \xi_0, E \rangle = \phi$. Hence, if $\xi_1 = \text{Proj}_E \xi_0$ then $\langle \xi_0, \xi_1 \rangle = \phi$. For $\xi \in S^{n-1} \cap E$, since $\xi - \text{Proj}_E\xi$ and $\xi_0 - \text{Proj}_E\xi_0$ are orthogonal, it follows that $\text{Proj}_{\xi_0} \xi = \text{Proj}_{\xi_0}(\text{Proj}_E \xi)$. Hence $\cos \angle(\xi, \xi_0) = \cos \angle(\xi, \xi_1) \cos \angle(\xi_1, \xi_0) = \cos \angle(\xi, \xi_1) \cos \phi$. Since the function $F$ is
Since, it is easy to see that:

\[ G(B) \text{ by using } G(\eta_0, E), \text{ and let } \tilde{g} = T(g). \text{ Then:} \]

\[ R_k(f)(\phi) = R_k(F)(E) = \int_{S^{n-1} \cap E} F(\xi) d\mu_E(\xi) = \int_{S^{n-1} \cap E} f(\angle(\xi, \xi_0)) d\mu_E(\xi) \]

\[ = \int_{S^{n-1} \cap E} f(\cos^{-1}(\cos \angle(\xi, \xi_1) \cos \phi)) d\mu_E(\xi) \]

\[ = c_k \int_0^{\pi/2} f(\cos^{-1}(\cos \phi \cos \theta)) \sin^{k-2} \theta d\theta. \]

Performing the change of variables \( t = \cos \theta, \phi = \cos \phi \) above, we immediately have:

**Corollary 3.2.** Let \( \tilde{f} \in C[0,1] \) and \( 2 \leq k \leq n - 1 \). Then:

\[ \tilde{R}_k(\tilde{f})(s) = c_k \int_0^1 \tilde{f}(st)(1 - t^2)^{k-3} dt, \]

where the value of \( c_k \) is the same as in Lemma 3.1.

Next, we introduce \( C_{\xi_0}(G(n, k)) \), the linear subspace of all functions in \( C(G(n, k)) \) invariant under the action of \( O_{\xi_0}(n-1) \). We refer to members of \( C_{\xi_0}(G(n, k)) \) as functions of revolution on the Grassmannian. As before, it is clear that \( G \in C_{\xi_0}(G(n, k)) \) iff \( G(E) = g(\angle(\xi_0, E)) \) for \( g \in C([0, \pi/2]) \). We have the following:

**Lemma 3.3.** Let \( G \in C_{\xi_0}(G(n, k)) \) such that \( G(E) = g(\angle(\xi_0, E)) \), and let \( \tilde{g} = T(g) \). Then:

\[ \int_{G(n,k)} G(E) d\eta_{n,k}(E) = b_{n,k} \int_0^{\pi/2} g(\phi) \sin^{n-k-1} \phi \cos^{k-1} \phi d\phi \]

\[ = b_{n,k} \int_0^1 \tilde{g}(s)(1 - s^2)^{\frac{n-k-2}{2}} s^{k-1} ds, \]

Where \( \eta_{n,k} \) is the Haar probability measure on \( G(n, k) \), and the value of \( b_{n,k} \) may be deduce by using \( G \equiv g \equiv \tilde{g} \equiv 1 \).

**Proof.** Clearly:

\[ \int_{G(n,k)} G(E) d\eta_{n,k}(E) = \int_0^{\pi/2} g(\phi) d\eta_{n,k} \{ E \in G(n,k); \angle(\xi_0, E) \leq \phi \} . \]

Since \( \sigma_n \) and \( \eta_{n,k} \) are rotation-invariant, it follows that \( \eta_{n,k} \{ E \in G(n,k); \angle(\xi_0, E) \leq \phi \} = \sigma_n \{ \xi \in S^{n-1}; \angle(\xi, E_0) \leq \phi \} \) for any \( E_0 \in G(n,k) \). Using bi-polar coordinates (e.g. [26, Chapter IX]), it is easy to see that:

\[ d\sigma_n \{ \xi \in S^{n-1}; \angle(\xi, E_0) \leq \phi \} = b_{n,k} \sin^{n-k-1} \phi \cos^{k-1} \phi d\phi, \]

for some \( b_{n,k} \). This concludes the proof of the first equality of the lemma, and the second one follows by the change of variables \( s = \cos(\phi) \).

Next, we find an expression for the dual spherical Radon-Transform of a function in \( C_{\xi_0}(G(n,k)) \). As before, it is clear that if \( F \in C_{\xi_0}(S^{n-1}) \) then \( R_k(F) \in C_{\xi_0}(G(n,k)) \), and that if \( G \in C_{\xi_0}(G(n,k)) \) then \( R_k^*(G) \in C_{\xi_0}(S^{n-1}) \). If \( G \in C_{\xi_0}(G(n,k)) \) is given by \( G(E) = g(\angle(\xi_0, E)) \), we denote by \( R_k^*(g) \in C([0, \pi/2]) \) the function given by \( R_k^*(g) = R_k(\angle(\xi_0, g)) = \)
$R^*_k(G)(\xi)$. As usual, we define $\tilde{R}^*_k : C[0,1] \to C[0,1]$ by $\tilde{R}^*_k = T \circ R^*_k \circ T^{-1}$. The standard duality relation:

$$\int_{S^{n-1}} R^*_k(G)(\xi) F(\xi) d\sigma_n(\xi) = \int_{G(n,k)} G(E) R_k(F)(E) d\eta_{n,k}(E)$$

is immediately translated using (3.1) and Lemma 3.3 into the following duality relation between $R_k$ and $R^*_k$ on $C([0,1])$:

**Lemma 3.4.** Let $\tilde{f}, \tilde{g} \in C([0,1])$ and $1 \leq k \leq n - 1$. Then:

$$\int_0^1 \tilde{R}^*_k(\tilde{g})(t) \tilde{f}(t)(1 - t^2)^{\frac{n-3}{2}} dt = d_{n,k} \int_0^1 \tilde{g}(s) \tilde{R}_k(\tilde{f})(s)(1 - s^2)^{\frac{n-k-2}{2}} s^{k-1} ds$$

where the value of $d_{n,k}$ is found by using $\tilde{f}, \tilde{g} \equiv 1$, in which case $\tilde{R}_k(\tilde{f}), \tilde{R}^*_k(\tilde{g}) \equiv 1$.

We can now deduce an expression for $\tilde{R}^*_k$:

**Lemma 3.5.** Let $\tilde{g} \in C([0,1])$ and $2 \leq k \leq n - 1$. Then:

$$\tilde{R}^*_k(\tilde{g})(t) = e_{n,k} \int_0^1 \tilde{g}(\sqrt{1 - s^2(1 - t^2)})(1 - s^2)^{\frac{k-3}{2}} s^{n-k-1} ds,$$

where the value of $e_{n,k}$ is found by using $\tilde{g} \equiv 1$, in which case $\tilde{R}^*_k(\tilde{g}) \equiv 1$.

**Proof.** We start with Lemma 3.4 and use the formula for $\tilde{R}_k$ given in Corollary 3.2:

$$\int_0^1 \tilde{R}^*_k(\tilde{g})(t) \tilde{f}(t)(1 - t^2)^{\frac{n-3}{2}} dt = d_{n,k} \int_0^1 \tilde{g}(s) \tilde{R}_k(\tilde{f})(s)(1 - s^2)^{\frac{n-k-2}{2}} s^{k-1} ds$$

$$= d_{n,k}c_k \int_0^1 \tilde{g}(s) \int_0^1 \tilde{f}(st)(1 - t^2)^{\frac{k-3}{2}} dt(1 - s^2)^{\frac{n-k-2}{2}} s^{k-1} ds$$

$$= d_{n,k}c_k \int_0^1 \tilde{f}(v) \int_v^1 \tilde{g}(s)(1 - v^2)^{\frac{k-3}{2}} (1 - s^2)^{\frac{n-k-2}{2}} s^{k-2} dsdv.$$  

Since this is true for any $\tilde{f} \in C([0,1])$, setting $e_{n,k} = d_{n,k}c_k$, we conclude that:

$$\tilde{R}^*_k(\tilde{g})(t) = e_{n,k}(1 - t^2)^{-\frac{n-3}{2}} \int_t^1 \tilde{g}(s) (1 - v^2)^{\frac{k-3}{2}} (1 - s^2)^{\frac{n-k-2}{2}} s^{k-2} ds.$$  

By the change of variable $s = \sqrt{1 - (s')^2(1 - t^2)}$, one easily checks that the assertion of the lemma is obtained. \qed

We now recall the definition of the “perp” operator $I$ from the Introduction, and extend it to the context of functions of revolution. For every $k = 1, \ldots, n - 1$, we define $I : C(G(n,k)) \to C(G(n,n-k))$ as $I(f)(E) = f(E^\perp)$ for all $E \in G(n,n-k)$, without specifying the index $k$. $I$ is obviously self-adjoint:

$$\int_{G(n,n-k)} I(F)(H)G(H)d\eta_{n-k}(H) = \int_{G(n,k)} F(E)I(G)(E)d\eta_k(E),$$

for all $F \in C(G(n,k))$ and $G \in C(G(n,n-k))$, where $\eta_m$ denotes the Haar probability measure on $G(n,m)$. 

2. GENERALIZED INTERSECTION BODIES ARE NOT EQUIVALENT
Since $\angle(\xi_0, E) = \pi/2 - \angle(\xi_0, E^\perp)$, it is clear that for $G \in C_{G_0}(G(n, k))$ such that $G(E) = g(\angle(\xi_0, E))$ for every $E \in G(n, k)$, $I(G)(H) = g(\pi/2 - \angle(\xi_0, H))$ for every $H \in G(n, n - k)$. We therefore define $I : C([0, \pi/2]) \to C([0, \pi/2])$ as $I(g)(\phi) = g(\pi/2 - \phi)$. Similarly, for $\tilde{g} \in C([0, 1])$, we define $I(\tilde{g})(s) = \tilde{g}(\sqrt{1 - s^2})$. Clearly, if $G(E) = \tilde{g}(\cos(\angle(\xi_0, E)))$ then $I(G)(H) = I(\tilde{g})(\cos(\angle(\xi_0, H)))$. Hence in both cases $I$ must be self-adjoint, and this can be also verified directly. As an immediate corollary of Lemma 3.5, we have:

**Corollary 3.6.** Let $\tilde{g} \in C([0, 1])$ and $2 \leq k \leq n - 1$. Then:

$$(I \circ R_k)^*(\tilde{g})(t) = e_{n,k} \int_0^1 \tilde{g}(s\sqrt{1 - t^2})(1 - s^2)^{\frac{k-3}{2}} s^{n-k-1} ds,$$

where the value of $e_{n,k}$ is the same as in Lemma 3.5.

We are now ready to construct the counter-example to Koldobsky’s question, as described in the next section.

### 4. The Construction

The main step in the proof of Theorem 1.2, is the following:

**Proposition 4.1.** For any $n \geq 4$, $2 \leq k \leq n - 2$ and $s_0 \in (0, 1)$, there exists an infinitely smooth function $\tilde{g} \in C([0, 1])$ such that:

1. For all $t \in [0, 1]$:
   $$\tilde{R}_{n-k}(\tilde{g})(t) = e_{n,k} \int_0^1 \tilde{g}(s\sqrt{1 - t^2})(1 - s^2)^{\frac{n-k-3}{2}} s^{k-1} ds \geq 1.$$

2. For all $t \in [0, 1]$:
   $$(I \circ R_k)^*(\tilde{g})(t) = e_{n,k} \int_0^1 \tilde{g}(s\sqrt{1 - t^2})(1 - s^2)^{\frac{k-3}{2}} s^{n-k-1} ds \geq 1.$$

3. $\tilde{g}(s_0) = -1$.

**Proof.** Let $\varepsilon > 0$ be such that $[s_0 - 2\varepsilon, s_0 + 2\varepsilon] \subset (0, 1)$. Let $T_t, T_t' \in C([0, 1])$ be defined by $T_t(s) = \sqrt{1 - s^2(1 - t^2)}$ and $T_t'(s) = s\sqrt{1 - t^2}$, and let $\lambda$ denote the Lebesgue measure on $\mathbb{R}$. It is elementary to check that the maximum of $\lambda \left\{ T_t^{-1}[s_0 - 2\varepsilon, s_0 + 2\varepsilon] \right\}$ over $t \in [0, 1]$ is attained at $t = s_0 - 2\varepsilon$, in which case it is equal to:

$$\delta_1 := \max_{t \in [0,1]} \lambda \left\{ T_t^{-1}[s_0 - 2\varepsilon, s_0 + 2\varepsilon] \right\} = 1 - \sqrt{1 - \frac{(s_0 + 2\varepsilon)^2}{1 - (s_0 - 2\varepsilon)^2}} < 1.$$

An analogous computation shows that the maximum of $\lambda \left\{ (T_t')^{-1}[s_0 - 2\varepsilon, s_0 + 2\varepsilon] \right\}$ over $t \in [0, 1]$ is attained at $t = \sqrt{1 - (s_0 + 2\varepsilon)^2}$, in which case it is equal to:

$$\delta_2 := \max_{t \in [0,1]} \lambda \left\{ (T_t')^{-1}[s_0 - 2\varepsilon, s_0 + 2\varepsilon] \right\} = \frac{4\varepsilon}{s_0 + 2\varepsilon} < 1.$$

Set $\delta := \max(\delta_1, \delta_2) < 1$. Now denote by $\mu_{n,m}$ the measure $e_{n,m}(1 - s^2)^{\frac{m-3}{2}} s^{n-m-1} ds$ on $[0, 1]$, for $2 \leq m \leq n - 2$. These are probability measures, as witnessed by using $\tilde{g} \equiv 1$ in
Lemma 3.5, in which case $\tilde{R}_k^*(\tilde{g}) \equiv 1$. Since their densities (with respect to $\lambda$) are absolutely continuous and do not vanish on $(0,1)$, a compactness argument shows that (fixing $n$):

$$\gamma := \sup_{v \in [0,1], 2 \leq m \leq n-2} \mu_{n,m}([v, v + \delta]) < 1.$$  

Set $\gamma^* = \frac{1+\gamma}{1-\gamma}$. We conclude by constructing $\tilde{g}$ as follows. Set $\tilde{g}(s) = -1$ for $s \in [s_0 - \varepsilon, s_0 + \varepsilon]$, $\tilde{g}(s) = \gamma^*$ for $s \in [0,1] \setminus [s_0 - 2\varepsilon, s_0 + 2\varepsilon]$, and for $s \in [s_0 - 2\varepsilon, s_0 + 2\varepsilon] \setminus [s_0 - \varepsilon, s_0 + \varepsilon]$ set $\tilde{g}(s) \in [-1, \gamma^*]$ so that the resulting function $\tilde{g} \in C[0,1]$ is in fact infinitely smooth (using standard methods). Alternatively, we could simply define $\tilde{g}(s) = (\gamma^* + 1)(s - s_0)^2 - 1$ on $[0,1]$. Setting:

$$\beta_1(t) := \mu_{n,n-k} \{ s \in [0,1]; T_i(s) \in [s_0 - 2\varepsilon, s_0 + 2\varepsilon] \},$$

the definition of $\gamma$ and $\delta$ imply that $\beta_1(t) \leq \gamma$ for all $t \in [0,1]$, hence:

$$\int_0^1 \tilde{g}(\sqrt{1 - s^2(1 - t^2)})d\mu_{n,n-k}(s) \geq \gamma^*(1 - \beta_1(t)) - \beta_1(t) \geq 1$$

for all $t \in [0,1]$. Similarly, setting:

$$\beta_2(t) := \mu_{n,k} \{ s \in [0,1]; T_i'(s) \in [s_0 - 2\varepsilon, s_0 + 2\varepsilon] \},$$

we have $\beta_2(t) \leq \gamma$ for all $t \in [0,1]$, and:

$$\int_0^1 \tilde{g}(s\sqrt{1 - t^2})d\mu_{n,k}(s) \geq \gamma^*(1 - \beta_2(t)) - \beta_2(t) \geq 1$$

for all $t \in [0,1]$. This concludes the proof. \hfill \Box

**Remark 4.1.** Note that for $k = 1$ and $k = n-1$ the above reasoning fails, as the measure $\mu_{n,1}$ is a singular measure.

**Remark 4.2.** Note also that the function $\tilde{g}$ we have constructed in fact satisfies the claims (1) and (2) for all values of $k$ in the range $2 \leq k \leq n-2$.

We can now almost conclude the proof of Theorem 1.2. We still need one last observation, since a-priori, the fact that $\tilde{g}(s_0) < 0$ does not guarantee that the function $G \in C(G(n,n-k))$ defined as $G(E) = \tilde{g}(\cos(\angle(\xi_0, E)))$, is not a non-negative functional on $R_{n-k}(C(S^{n-1}))$. This is resolved by the following:

**Lemma 4.2.** The polynomials on $[0,1]$ are in the range of $\tilde{R}_{n-k}(C([0,1]))$.

**Proof.** This is immediate by Corollary 3.2, because if $\tilde{\rho}(t) = t^m$ ($m \geq 0$), then:

$$\tilde{R}_k(\tilde{\rho})(s) = c_k \int_0^1 \tilde{\rho}(st)(1 - t^2)^{\frac{k-3}{2}} dt = d_{k,m}s^m,$$

with $d_{k,m} > 0$. Hence polynomials are mapped to polynomials by $\tilde{R}_{n-k}$, and any polynomial in the range may be obtained. \hfill \Box

By the Weierstrass approximation theorem, it follows that:

**Corollary 4.3.** The range of $\tilde{R}_{n-k}$ is dense in $C([0,1])$.

We can now turn to the proof of Theorem 1.2.
Proof of Theorem 1.2. Let $\tilde{g} \in C[0,1]$ be the infinitely smooth function constructed in Proposition 4.1, with, say $s_0 = 1/2$. Fix some $\xi_0 \in S^{n-1}$, and let $G \in C_{\xi_0}(G(n,n-k))$ be defined by $G(E) = \tilde{g}(\cos(\langle \xi_0, E \rangle))$ for every $E \in G(n,n-k)$. Since the functions $\tilde{g}$, $\cos$ and $\angle(\xi, \cdot)$ are infinitely smooth on their corresponding domains, so is their composition, hence $G$ is infinitely smooth on $G(n,n-k)$. By the construction of $\tilde{g}$ and the compatibility of $R^*_{n-k}$ and $(I \circ R_k)^*$ with $\tilde{R}^*_{n-k}$ and $(I \circ \tilde{R}_k)^*$, respectively, it follows that $R^*_{n-k}(G) = \tilde{R}^*_{n-k}(\tilde{g}) \geq 1$ and $(I \circ R_k)^*(G) = (I \circ \tilde{R}_k)^*(\tilde{g}) \geq 1$. It remains to show that $G$ is not a non-negative functional on $R_{n-k}(C(S^{n-1}))$. Let $H \in C_{\xi_0}(S^{n-1})$ be such that $H(\xi) = \tilde{h}(\cos(\angle(\xi_0, \xi)))$ for some $\tilde{h} \in C([0,1])$. Then by Lemma 3.3:

\begin{equation}
\int_{G(n,n-k)} G(E)R_{n-k}(H)(E) d\eta_{n,k} = \int_0^1 \tilde{g}(s)\tilde{R}_{n-k}(\tilde{h})(s)(1-s^2)^{-\frac{n-k-2}{2}} s^{k-1} ds.
\end{equation}

Since $\tilde{g}(s)(1-s^2)^{-\frac{n-k-2}{2}} s^{k-1}$ is a continuous function on $[0,1]$ whose value at $s_0$ is negative, by Corollary 4.3 we can find a function $\tilde{h} \in C([0,1])$ such that the integral in (4.1) is negative. This concludes the proof. 

5. Additional formulations

In this section, we give several additional equivalent formulations to the main result of this note, using the language of Fourier transforms of homogeneous distributions (we refer the reader to [18] for more on this subject).

We denote by $S'(\mathbb{R}^n)$ the space of rapidly decreasing infinitely differentiable test functions in $\mathbb{R}^n$, and by $S'(\mathbb{R})$ the space of distributions over $S'(\mathbb{R})$. The Fourier transform $\hat{f}$ of a distribution $f \in S'(\mathbb{R}^n)$ is defined by $\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle$ for every test function $\phi$, where $\hat{\phi}(y) = \int \phi(x) \exp(-i(x,y))dx$. A distribution $f$ is called homogeneous of degree $p \in \mathbb{R}$ if $\langle f, \phi(\cdot/t) \rangle = |t|^{n+p} \langle f, \phi \rangle$ for every $t > 0$, and it is called even if the same is true for $t = -1$. An even distribution $f$ always satisfies $\langle \hat{f}, \phi \rangle = (2\pi)^n f$. The Fourier transform of an even homogeneous distribution of degree $p$ is an even homogeneous distribution of degree $-n - p$. A distribution $f$ is called positive if $\langle f, \phi \rangle \geq 0$ for every $\phi \geq 0$, implying that $f$ is necessarily a non-negative Borel measure on $\mathbb{R}^n$. We use Schwartz’s generalization of Bochner’s Theorem ([9]) as a definition, and call a homogeneous distribution positive-definite if its Fourier transform is a positive distribution.

5.1. Embeddings in $L_p$. Recall the following:

Definition. A normed space $(\mathbb{R}^n, \|\|)$ is said to embed in $L_p$ ($p \geq 1$) iff there exists a basis $x_1, \ldots, x_n \in \mathbb{R}^n$ and functions $f_1, \ldots, f_n \in L_p([0,1])$ such that $\|\sum_{i=1}^n a_i x_i\|^p = \int |\sum_{i=1}^n a_i f_i(t)|^p dt$ for all scalars $\{a_i\}$.

This definition may be extended to the range $0 < p < 1$, in which case $\|\| \|$ is no longer necessarily a norm, but rather the Minkowski functional of some star-body. In addition, the following equivalent definition is known (e.g. [17]). Note that this definition makes sense for $p > -1$ (and $p \neq 0$, the case $p = 0$ requires passing to the limit).

Equivalent Definition. $(\mathbb{R}^n, \|\|)$ embeds in $L_p$ ($p > -1$, $p \neq 0$) iff

\begin{equation}
\|x\|^p = \int_{S^{n-1}} |\langle x, \theta \rangle|^p d\mu(\theta),
\end{equation}
for some $\mu \in \mathcal{M}_+(S^{n-1})$, the cone of non-negative Borel measures on $S^{n-1}$.

Unfortunately, this characterization breaks down at $p = -1$ since the above integral no longer converges. However, A. Koldobsky showed that it is possible to regularize this integral by using Fourier-transforms of distributions, and gave the following definition in [17]:

**Definition.** $(\mathbb{R}^n, \|\cdot\|)$ embeds in $L_{-p}$ for $0 < p < n$ iff there exists a measure $\mu \in \mathcal{M}_+(S^{n-1})$ such that for any even test-function $\phi$:

$$\int_{\mathbb{R}^n} \|x\|^{-p} \phi(x) dx = \int_{S^{n-1}} \int_0^\infty t^{p-1} \hat{\phi}(t\theta) dt d\mu(\theta).$$

Consequently, the following characterization was given in [17]:

**Theorem 5.1** (Koldobsky). The following are equivalent for a centrally-symmetric starbody $K$ in $\mathbb{R}^n$:

1. $K \in \mathcal{I}_n^k$.
2. $\|\cdot\|^{-k}_K$ is a positive definite distribution on $\mathbb{R}^n$.
3. The space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in $L_{-k}$.

In addition to the characterization (3) in Theorem 5.1 of $\mathcal{I}_n^k$ as the class of unit-balls of subspaces of scalar $L_{-k}$ spaces, a functional analytic characterization of $\mathcal{BP}_n^k$ as the class of unit-balls of subspaces of certain vector-valued $L_{-k}$ spaces was given in [17]. To explain this better, we state the definition given by Koldobsky:

**Definition.** $(\mathbb{R}^n, \|\cdot\|)$ embeds in $L_{-p}(\mathbb{R}^k)$ for $0 < p < n$ iff there exists a measure $\mu \in \mathcal{M}_+(\mathbb{R}^{nk})$ such that for any even test-function $\phi$:

$$\int_{\mathbb{R}^n} \|x\|^{-p} \phi(x) dx = \int_{\mathbb{R}^{nk}} \int_{\mathbb{R}^k} \|v\|^{p-k}_2 \hat{\phi}\left(\sum_{i=1}^k v_i \xi_i\right) dvd\mu(\xi).$$

For $k = 1$ it is easy to see that this coincides with the definition of embedding in $L_{-p}$. Using this definition, the following was shown in [17]:

**Theorem 5.2** (Koldobsky). $K \in \mathcal{BP}_n^k$ iff $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in $L_{-k}(\mathbb{R}^k)$.

For $p > 0$, it is known that every separable vector valued $L_p$ space is isometric to a subspace of a scalar $L_p$ space and vice-versa. Translating Theorem 1.1 into the language of $L_p$ spaces, we see that this is no longer true when $p = -k$, $2 \leq k \leq n - 2$:

**Corollary 5.3.** Let $n \geq 4$ and $2 \leq k \leq n - 2$. Then there exists an infinitely smooth centrally-symmetric body of revolution $K$ such that $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in $L_{-k}$ but does not embed in $L_{-k}(\mathbb{R}^k)$.

### 5.2. Non trivial spaces which embed in $L_p$ ($p < -1$)

We proceed to describe another property of $L_p$ spaces which breaks down when passing the critical value of $p = -1$.

**Definition.** Let $SL_p^n (p \neq 0)$ denote the set of all star-bodies $K$ in $\mathbb{R}^n$ for which $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in $L_p$. 
For \( p \neq 0 \), let the \( p\)-norm sum of two bodies \( L_1, L_2 \) be defined as the body \( L \) satisfying \( \| · \|_L^p = \| · \|_{L_1}^p + \| · \|_{L_2}^p \). Obviously, the \( p\)-norm sum coincides with the \((−p)\)-radial sum, defined in the introduction (before the Structure Theorem).

**Definition.** Let \( D^n_p \) \( (p \neq 0) \) denote the class of bodies created from the Euclidean ball \( D_n \) by applying full-rank linear-transformations, \( p\)-norm sums, and taking the limit in the radial metric \( d_r \).

Using the characterization in (5.1), it is easy to show (e.g. \([12, \text{Theorem } 6.13]\)) that for \( p > −1 \ (p \neq 0) \), \( SL^n_p = D^n_p \). In order to understand what happens when \( p \leq −1 \), we turn to the following characterization of \( \mathcal{BP}^n_k \), first proved by Goodey and Weil in \([11]\) for intersection-bodies (the case \( k = 1 \)), and extended to general \( k \) by Grinberg and Zhang in \([12]\):

**Theorem 5.4** (Grinberg and Zhang). \( \mathcal{BP}^n_k = D^n_{−k} \) for \( k = 1, \ldots, n − 1 \).

Recall that \( T^n_1 = \mathcal{BP}^n_1 \) is the class of all intersection-bodies in \( \mathbb{R}^n \) and \( T^n_{n−1} = \mathcal{BP}^n_{n−1} \) is the class of all centrally-symmetric star-bodies in \( \mathbb{R}^n \) (this is clear from the definitions, see also the Structure Theorem from the introduction). Since \( T^n_k = SL^n_{−k} \) by characterization (3) of Theorem 5.1, we see that \( SL^n_{−k} = D^n_{−k} \) for \( k = 1 \) and \( k = n − 1 \). However, Theorem 1.1 implies that this is no longer true for \( 2 \leq k \leq n − 2 \).

**Corollary 5.5.** Let \( n \geq 4 \) and \( 2 \leq k \leq n − 2 \). Then \( SL^n_{−k} \setminus D^n_{−k} \neq \emptyset \).

Note that since \( \mathcal{BP}^n_k \subset T^n_k \), it is always true that \( D^n_{−k} \subset SL^n_{−k} \) (in fact, this is straightforward to check directly, implying that \( \mathcal{BP}^n_k \subset T^n_k \) by using Theorems 5.1 and 5.4). In some sense, the members of \( D^n_{−k} \) are the “trivial” elements of \( SL^n_{−k} \), since obviously \( D_n \in SL^n_{−k} \), and \( SL^n_{−k} \) is closed under taking full-rank linear transformations, \((−k)\)-norm sums and and limit in the radial-metric. Corollary 5.5 should therefore be interpreted as saying that there are also “non-trivial” elements in \( SL^n_{−k} \), for \( 2 \leq k \leq n − 2 \).

5.3. **Non-trivial positive-definite homogeneous distributions.** We conclude by translating Corollary 5.5 into the language of Fourier transforms of homogeneous distributions.

**Notation.** Given an even \( f \in C(S^{n−1}) \), we denote by \( E_p(f) \) its homogeneous extension of degree \( p \) onto \( \mathbb{R}^n \) (formally excluding \( \{0\} \) if \( p < 0 \)), i.e. \( E_p(f)(t\theta) = t^pf(\theta) \) for \( t > 0 \) and \( \theta \in S^{n−1} \). We denote by \( E_p^\wedge(f) \) the Fourier transform of \( E_p(f) \) as a distribution. Given a full-rank linear transformation \( T \) on \( \mathbb{R}^n \), we denote \( T(E_p(f)) = E_p(f) \circ T^{-1} \).

Note that \( E_p^\wedge(f) \) need not necessarily be a continuous function on \( \mathbb{R}^n \setminus \{0\} \), nor even a measure on \( \mathbb{R}^n \). In order to ensure that \( E_p^\wedge(f) \) is a continuous function, we need to add some smoothness assumptions on \( f \) \((\text{[18]}\)) We remark that for an infinitely smooth function \( f \in C(S^{n−1}) \), \( E_p^\wedge(f) \) is infinitely smooth on \( \mathbb{R}^n \setminus \{0\} \) for any \( p \in (−n, 0) \). Whenever \( E_p^\wedge(f) \) is continuous on \( \mathbb{R}^n \setminus \{0\} \), it is uniquely determined by its value on \( S^{n−1} \) (by homogeneity), so we identify (abusing notation) between \( E_p^\wedge(f) \) and its restriction to \( S^{n−1} \). Clearly \( E_{−k}(\rho^n_K) = ||·||_K^k \) for a star-body \( K \), hence \( T(E_{−k}(\rho^n_K)) = E_{−k}(\rho^n_{T(K)}) \). Again, we identify (by homogeneity) between \( T(E_p(f)) \) and its restriction on \( S^{n−1} \).

It is easy to check (e.g. \([22]\)) that for any infinitely smooth \( K \in D^n_{−k} \), we have \( E_{−k}^\wedge(\rho^n_K) \geq 0 \) (and clearly \( \rho^n_K \geq 0 \)). In fact, this immediately follows from the fact that this is true
for $D_n \in D_n^\odot k$, the linearity of the Fourier transform, and its behavior under full-rank linear transformations. With Theorem 5.4 and characterization (2) of Theorem 5.1 in mind, asking whether $\mathcal{BP}_k^n = T_k^n$ is equivalent to asking whether the only infinitely smooth functions $f \in C(S^{n-1})$ such that $f \geq 0$ and $E_{-k}^\wedge(f) \geq 0$, are the ones such that $f = \rho_k^K$ for some $K \in D_n^\odot k$. In other words, whether every such $f$ can be approximated (in the maximum norm in $C(S^{n-1})$, which is clearly the same for $f$ and for $f^{1/k}$) by functions of the form $\sum_{i=1}^n T_i(E_{-k}(1))$, where $T_i$ are full-rank linear transformations. The following is thus an immediate consequence of Theorem 1.1:

**Corollary 5.6.** Let $n \geq 4$ and $2 \leq k \leq n - 2$. Then there exists a “non-trivial” infinitely smooth function of revolution $f \in C(S^{n-1})$ such that $f \geq 0$ and $E_{-k}^\wedge(f) \geq 0$. By “non-trivial”, we mean that $f$ cannot be approximated in the maximum norm on $C(S^{n-1})$ by functions of the form $\sum_{i=1}^n T_i(E_{-k}(1))$, where $\{T_i\}$ are full-rank linear transformations in $\mathbb{R}^n$.

5.4. **Concluding Remark.** To conclude, we comment that although the original definitions of $\mathcal{BP}_k^n$ and $T_k^n$ make sense only for integer values of $k$ (between 1 and $n-1$), some of the alternative characterizations of these classes stated in this section make sense for arbitrary real-valued $k$, for $0 < k < n$. In particular, characterizations (2) and (3) of Theorem 5.1 for the class $T_k^n$ and Theorem 5.4 for the class $\mathcal{BP}_k^n$ may be taken as definitions for these classes of star-bodies in this extended range of $k$. It then makes sense to ask whether Theorem 1.1 also holds for any non-integer $1 < k < n - 1$. Although we do not proceed in this direction, the answer should be positive, since our construction of the function $\tilde{g}$ in Proposition 4.1 is purely analytic, and everything still works for arbitrary real-valued $k$, for $1 < k < n$.

**References**

10. A. A. Giannopoulos, *A note on a problem of h. busemann and c.m. petty concerning sections of symmetric convex bodies*, Mathematika **37** (1990), 239–244.

E-mail address: emanuel.milman@weizmann.ac.il

Department of Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel.
CHAPTER 3

A COMMENT ON THE LOW-DIMENSIONAL BUSEMANN-PETTY PROBLEM

EMANUEL MILMAN

Lecture Notes in Math. 1910, GAFA Seminar Notes 2004-5, 245-253

Abstract. The generalized Busemann-Petty problem asks whether centrally-symmetric convex bodies having larger volume of all $m$-dimensional sections necessarily have larger volume. When $m > 3$ this is known to be false, but the cases $m = 2, 3$ are still open. In those cases, it is shown that when the smaller body’s radial function is a $n-m$-th root of the radial function of a convex body, the answer to the generalized Busemann-Petty problem is positive (for any larger star-body). Several immediate corollaries of this observation are also discussed.

1. Introduction

Let $\text{Vol}(L)$ denote the Lebesgue measure of a set $L \subset \mathbb{R}^n$ in its affine hull, and let $G(n, k)$ denote the Grassmann manifold of $k$ dimensional subspaces of $\mathbb{R}^n$. Let $D_n$ denote the Euclidean unit ball, and $S^{n-1}$ the Euclidean sphere. All of the bodies considered in this note will be assumed to be centrally symmetric star-bodies, defined by a continuous radial function $\rho_K(\theta) = \max\{r \geq 0 \mid r\theta \in K\}$ for $\theta \in S^{n-1}$ and a star-body $K$.

The Busemann-Petty problem, first posed in [4], asks whether two centrally-symmetric convex bodies $K$ and $L$ in $\mathbb{R}^n$ satisfying:

\begin{equation}
\text{Vol}(K \cap H) \leq \text{Vol}(L \cap H) \quad \forall H \in G(n, n-1)
\end{equation}

necessarily satisfy $\text{Vol}(K) \leq \text{Vol}(L)$. For a long time this was believed to be true (this is certainly true for $n = 2$), until a first counterexample was given in [16] for a large value of $n$. In the same year, the notion of an intersection-body was first introduced by Lutwak in [17] (see also [18] and Section 2 for definitions) in connection to the Busemann-Petty problem. It was shown in [18] (and refined in [5]) that the answer to the Busemann-Petty problem is equivalent to whether all convex bodies in $\mathbb{R}^n$ are intersection bodies. Subsequently, it was shown in a series of results ([16], [1], [2], [8], [21], [5], [6], [14], [25], [7]), that this is true for $n \leq 4$, but false for $n \geq 5$.

In [24], Zhang considered a natural generalization of the Busemann-Petty problem, which asks whether two centrally-symmetric convex bodies $K$ and $L$ in $\mathbb{R}^n$ satisfying:

\begin{equation}
\text{Vol}(K \cap H) \leq \text{Vol}(L \cap H) \quad \forall H \in G(n, n-k)
\end{equation}

Supported in part by BSF and ISF.
necessarily satisfy $\text{Vol}(K) \leq \text{Vol}(L)$, where $k$ is some integer between 1 and $n - 1$. Zhang showed that the generalized $k$-codimensional Busemann-Petty problem is also naturally associated to another class of bodies, which will be referred to as $k$-Busemann-Petty bodies (note that these bodies are referred to as $n - k$-intersection bodies in [24] and generalized $k$-intersection bodies in [15]), and that the generalized $k$-codimensional problem is equivalent to whether all convex bodies in $\mathbb{R}^n$ are $k$-Busemann-Petty bodies. Analogously to the original problem, it was shown in [24] that if $K$ and $L$ are two centrally-symmetric star-bodies (not necessarily convex) satisfying (1.2), and if $K$ is a $k$-Busemann-Petty body, then $\text{Vol}(K) \leq \text{Vol}(L)$.

It was shown in [3] (see also a correction in [22]), and later in [15], that the answer to the generalized $k$-codimensional problem is negative for $k < n - 3$, but the cases $k = n - 3$ and $k = n - 2$ still remain open (the case $k = n - 1$ is obviously true). Several partial answers to these cases are known. It was shown in [24] (see also [22]) that when $K$ is a centrally-symmetric convex body of revolution then the answer is positive for the pair $K, L$ with $k = n - 2, n - 3$ and any star-body $L$. When $k = n - 2$, it was shown in [3] that the answer is positive if $L$ is a Euclidean ball and $K$ is convex and sufficiently close to $L$. Several other generalizations of the Busemann-Petty problem were treated in [22]. Our main observation in this note concerns the cases $k = n - 2, n - 3$ and reads as follows:

**Theorem 1.1.** Let $K$ denote a centrally-symmetric convex body in $\mathbb{R}^n$. For $a = 2, 3$, let $K_a$ be the star-body defined by $\rho_{K_a} = \rho_K^{1/(n-a)}$. Then $K_a$ is a $(n-a)$-Busemann-Petty body, implying a positive answer to the $(n-a)$-codimensional Busemann-Petty problem (1.2) for the pair $K_a, L$ for any star-body $L$.

The case $a = 1$ is also true, but follows trivially since it is easy to see (e.g. [19]) that any star-body is an $n - 1$-Busemann-Petty body. The case $a = 2$ follows from $a = 3$ by a general result from [19], stating that if $K$ is a $k$-Busemann-Petty body and $L$ is given by $\rho_L = \rho_K^{k/l}$ for $1 \leq k < l \leq n - 1$, then $L$ is a $l$-Busemann-Petty body.

Theorem 1.1 has several interesting consequences. The first one is the following complementary result to the one aforementioned from [3]. Roughly speaking, it states that any small enough perturbation $K$ of the Euclidean ball, for which we have control over the second derivatives of $\rho_K$, satisfies the low-dimensional generalized Busemann-Petty problem (1.2) with any star-body $L$.

**Corollary 1.2.** For any $n$, there exists a function $\gamma : [0, \infty) \to (0, 1)$, such that the following holds: let $\varphi$ denote a twice continuously differentiable function on $S^{n-1}$ such that:

$$\max_{\theta \in S^{n-1}} |\varphi(\theta)| \leq 1, \quad \max_{\theta \in S^{n-1}} |\varphi_i(\theta)| \leq M, \quad \max_{\theta \in S^{n-1}} |\varphi_{i,j}(\theta)| \leq M,$$

for every $i, j = 1, \ldots, n - 1$, where $\varphi_i$ and $\varphi_{i,j}$ denote the first and second partial derivatives of $\varphi$ (w.r.t. any local coordinate system of $S^{n-1}$), respectively. Then the star-body $K^\epsilon$ defined by $\rho_{K^\epsilon} = 1 + \epsilon \varphi$ for any $|\epsilon| < \gamma(M)$ is a $(n-a)$-Busemann-Petty body for $a = 2, 3$, implying a positive answer to the $(n-a)$-codimensional Busemann-Petty problem (1.2) for $K^\epsilon$ and any star-body $L$.

Note that the definition of $K_a$ in Theorem 1.1 is highly non-linear with respect to $K$. Since the class of $k$-Busemann-Petty bodies is closed under certain natural operations (see
[19] for the latest known results), we can take advantage of this fact to strengthen the result of Theorem 1.1. For instance, it is well known (e.g. [10], [19]) that the class of $k$-Busemann-Petty bodies is closed under taking $k$-radial sums. The $k$-radial sum of two star-bodies $L_1, L_2$ is defined as the star-body $L$ satisfying $\rho^k_L = \rho^k_{L_1} + \rho^k_{L_2}$. When $k = 1$ this operation will simply be referred to as radial sum. The space of star-bodies in $\mathbb{R}^n$ is endowed with the natural radial metric $d_r$, defined as $d_r(L_1, L_2) = \max_{\theta \in S^{n-1}} |\rho_{L_1}(\theta) - \rho_{L_2}(\theta)|$. We will denote by $\mathcal{RC}^n$ the closure in the radial metric of the class of all star-bodies in $\mathbb{R}^n$ which are finite radial sums of centrally-symmetric convex bodies. It should then be clear that:

**Corollary 1.3.** Theorem 1.1 holds for any $K \in \mathcal{RC}^n$.

Our last remark in this note is again an immediate consequence of Theorem 1.1 and the following characterization of $k$-Busemann-Petty bodies due to Grinberg and Zhang ([10]), which generalizes the characterization of intersection-bodies (the case $k = 1$) given by Goodey and Weil ([9]):

**Theorem (Grinberg and Zhang).** A star-body $K$ is a $k$-Busemann-Petty body iff it is the limit of $\{K_i\}$ in the radial metric $d_r$, where each $K_i$ is a finite $k$-radial sums of ellipsoids $\{\mathcal{E}_j\}$,

$$\rho^k_{K_i} = \sum_{j=1}^m \rho^k_{\mathcal{E}_{i_j}}.$$

Applying Grinberg and Zhang’s Theorem to the bodies $K_n$ from Theorem 1.1, we immediately have:

**Corollary 1.4.** Let $K$ denote a centrally-symmetric convex body in $\mathbb{R}^n$. Then for $a = 2, 3$, $K$ is the limit in the radial metric $d_r$ of star-bodies $K_i$ having the form:

$$\rho^k_{K_i} = \sum_{j=1}^m \rho^{n-a}_{\mathcal{E}_{i_j}},$$

where $\{\mathcal{E}_j\}$ are ellipsoids.

2. Definitions and notations

A star body $K$ is said to be an intersection body of a star body $L$, if $\rho_K(\theta) = \text{Vol}(L \cap \theta^\perp)$ for every $\theta \in S^{n-1}$. $K$ is said to be an intersection body, if it is the limit in the radial metric $d_r$ of intersection bodies $\{K_i\}$ of star bodies $\{L_i\}$, where $d_r(K_1, K_2) = \sup_{\theta \in S^{n-1}} |\rho_{K_1}(\theta) - \rho_{K_2}(\theta)|$. This is equivalent (e.g. [18], [5]) to $\rho_K = R^*(d\mu)$, where $\mu$ is a non-negative Borel measure on $S^{n-1}$, $R^*$ is the dual transform (as in (2.1)) to the Spherical Radon Transform $R: C(S^{n-1}) \to C(S^{n-1})$, which is defined for $f \in C(S^{n-1})$ as:

$$R(f)(\theta) = \int_{S^{n-1} \cap \theta^\perp} f(\xi)d\sigma_{n-1}(\xi),$$

where $\sigma_{n-1}$ the Haar probability measure on $S^{n-2}$ (and we have identified $S^{n-2}$ with $S^{n-1} \cap \theta^\perp$).

Before defining the class of $k$-Busemann-Petty bodies we shall need to introduce the $m$-dimensional Spherical Radon Transform, acting on spaces of continuous functions as
follows:

\[ R_m : C(S^{n-1}) \longrightarrow C(G(n, m)) \]

\[ R_m(f)(E) = \int_{S^{n-1} \cap E} f(\theta)d\sigma_m(\theta), \]

where \( \sigma_m \) is the Haar probability measure on \( S^{m-1} \) and we have identified \( S^{m-1} \) with \( S^{n-1} \cap E \). The dual transform is defined on spaces of signed Borel measures \( \mathcal{M} \) by:

\[ R^*_m : \mathcal{M}(G(n, m)) \longrightarrow \mathcal{M}(S^{n-1}) \]

\[ \int_{S^{n-1}} fR^*_m(d\mu) = \int_{G(n, m)} R_m(f)d\mu \quad \forall f \in C(S^{n-1}), \]

and for a measure \( \mu \) with continuous density \( g \), the transform may be explicitly written in terms of \( g \) (see [24]):

\[ R^*_mg(\theta) = \int_{\theta \in E \in G(n, m)} g(E)d\nu_m(E), \]

where \( \nu_m \) is the Haar probability measure on \( G(n - 1, m - 1) \).

We shall say that a body \( K \) is a \( k \)-Busemann-Petty body if \( \rho^K_\mu = R^*_{m-k}(d\mu) \) as measures in \( \mathcal{M}(S^{n-1}) \), where \( \mu \) is a non-negative Borel measure on \( G(n, n - k) \). We shall denote the class of such bodies by \( \mathcal{BP}^n_k \). Choosing \( k = 1 \), for which \( G(n, n - 1) \) is isometric to \( S^{n-1}/\mathbb{Z}_2 \) by mapping \( H \) to \( S^{n-1} \cap H^\perp \), and noticing that \( R \) is equivalent to \( R_{n-1} \) under this map, we see that \( \mathcal{BP}^n_1 \) is exactly the class of intersection bodies.

We will also require, although indirectly, several notions regarding Fourier transforms of homogeneous distributions. We denote by \( \mathcal{S}(\mathbb{R}^n) \) the space of rapidly decreasing infinitely differentiable test functions in \( \mathbb{R}^n \), and by \( \mathcal{S}'(\mathbb{R}^n) \) the space of distributions over \( \mathcal{S}(\mathbb{R}^n) \). The Fourier Transform \( \hat{f} \) of a distribution \( f \in \mathcal{S}'(\mathbb{R}^n) \) is defined by \( \langle \hat{f}, \phi \rangle = \langle f, \phi \rangle \) for every test function \( \phi \), where \( \hat{\phi}(y) = \int \phi(x)\exp(-ix \cdot y)dx \). A distribution \( f \) is called homogeneous of degree \( p \in \mathbb{R} \) if \( \langle f, \phi(\cdot/t) \rangle = |t|^{n+p} \langle f, \phi \rangle \) for every \( t > 0 \), and it is called even if the same is true for \( t = -1 \). An even distribution \( f \) always satisfies \( \langle \hat{f}, \phi \rangle = (2\pi)^n f \). The Fourier Transform of an even homogeneous distribution of degree \( p \) is an even homogeneous distribution of degree \( -n - p \).

We will denote the space of continuous functions on the sphere by \( C(S^{n-1}) \). The spaces of even continuous and infinitely smooth functions will be denoted \( C_c(S^{n-1}) \) and \( C^\infty(S^{n-1}) \), respectively.

For a star-body \( K \) (not necessarily convex), we define its Minkowski functional as \( \|x\|_K = \min \{ t \geq 0 \mid x/t \in K \} \). When \( K \) is a centrally-symmetric convex body, this of course coincides with the natural norm associated with it. Obviously \( \rho_K(\theta) = \|\theta\|_K^{-1} \) for \( \theta \in S^{n-1} \).

3. PROOFS OF THE STATEMENTS

Before we begin, we shall need to recall several known facts about the Spherical Radon Transform \( R \), and its connection to the Fourier transform of homogeneous distributions. It is well known (e.g. [11, Chapter 3]) that \( R : C_c(S^{n-1}) \rightarrow C_c(S^{n-1}) \) is an injective operator, and that it is onto a dense set in \( C_c(S^{n-1}) \) which contains \( C^\infty_c(S^{n-1}) \). The connection with Fourier transforms of homogeneous distributions was demonstrated by Koldobsky, who showed (e.g. [13]) the following:
Lemma 3.1. Let $L$ denote a star-body in $\mathbb{R}^n$. Then for all $\theta \in S^{n-1}$:

$$\left(\|\cdot\|_L^{-n+1}\right)^{\wedge}(\theta) = \pi(n-1)\text{Vol}(D_{n-1}) R(\|\cdot\|_L^{-n+1})(\theta).$$

In particular $\left(\|\cdot\|_L^{-n+1}\right)^{\wedge}$ is continuous, and of course homogeneous of degree $-1$. Hence, if we denote $\rho_K(\theta) = \|\theta\|^\perp_K = \left(\|\cdot\|_L^{-n+1}\right)^{\wedge}(\theta)$ for $\theta \in S^{n-1}$ and use $\left(\|\cdot\|_K^{-1}\right)^{\wedge}(\theta) = (2\pi)^n \|\theta\|_L^{-n+1}$, we immediately get the following inversion formula for the Spherical Radon transform:

Lemma 3.2. Let $K$ denote a star-body in $\mathbb{R}^n$ such that $\rho_K$ is in the range of the Spherical Radon Transform. Then for all $\theta \in S^{n-1}$:

$$R^{-1}(\rho_K)(\theta) = \pi(n-1)\text{Vol}(D_{n-1}) (\|\cdot\|_K^{-1})^{\wedge}(\theta).$$

Koldobsky also discovered the following property of the Fourier transform of a norm of a convex body ([15, Corollary 2]):

Lemma 3.3. Let $K$ be an infinitely smooth centrally-symmetric convex body in $\mathbb{R}^n$. Then for every $E \in G(n,k)$:

$$\int_{S^{n-1} \cap E} \left(\|\cdot\|_K^{-n+k-1}\right)^{\wedge}(\theta)d\theta \geq 0.$$

Since $C_c^\infty(S^{n-1})$ is in the range of the Spherical Radon Transform, applying Lemma 3.3 with $k = n-3$ and using Lemma 3.2, we have:

Proposition 3.4. Let $K$ be an infinitely smooth centrally-symmetric convex body in $\mathbb{R}^n$. Then for every $E \in G(n,n-3)$:

$$\int_{S^{n-1} \cap E} R^{-1}(\rho_K)(\theta)d\theta \geq 0.$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. First, assume that $K$ is infinitely smooth and fix $\theta \in S^{n-1}$. Denote by $H_\theta \in G(n,n-1)$ the hyperplane $\theta^\perp$, and let $\sigma_{H_\theta}$ denote the Haar probability measure on $S^{n-1} \cap H_\theta$. Let $\eta_{H_\theta}$ denote the Haar probability measure on the homogeneous space $G^H_\theta(n,n-3) := \{E \in G(n,n-3)|E \subset H_\theta\}$, and let $\sigma_{H_\theta}$ denote the Haar probability measure on $S^{n-1} \cap E$ for $E \in G(n,n-3)$. Then:

$$\rho_K(\theta) = R(R^{-1}(\rho_K))(\theta) = \int_{S^{n-1} \cap H_\theta} R^{-1}(\rho_K)(\xi)d\sigma_{H_\theta}(\xi)$$

(3.1)

$$= \int_{E \in G^H_\theta(n,n-3)} \int_{S^{n-1} \cap E} R^{-1}(\rho_K)(\xi)d\sigma_{E}(\xi)d\eta_{H_\theta}(E).$$

The last transition is explained by the fact that the measure $d\sigma_{E}(\xi)d\eta_{H_\theta}(E)$ is invariant under orthogonal transformations preserving $H_\theta$, so by the uniqueness of the Haar probability measure, it must coincide with $d\sigma_{H_\theta}(\xi)$. Denoting:

$$g(F) = \int_{S^{n-1} \cap F^\perp} R^{-1}(\rho_K)(\xi)d\sigma_{E}(\xi)$$

for $F \in G(n,3)$, we see by Proposition 3.4 that $g \geq 0$. Plugging the definition of $g$ in (3.1), we have:

$$\rho_K(\theta) = \int_{E \in G^H_\theta(n,n-3)} g(E^\perp)d\eta_{H_\theta}(E) = \int_{F \in G_\theta(n,3)} g(F)d\nu_{\theta}(F),$$
where \( \nu_0 \) is the Haar probability measure on the homogeneous space \( G_\theta(n, 3) := \{ F \in G(n, 3) | \theta \in F \} \) and the transition is justified as above. By (2.2), we conclude that \( \rho_K = R^*_a(h) \) with \( g \geq 0 \), implying that the body \( K_3 \) satisfying \( \rho_{K_3}^{n-3} = \rho_K \) is in \( \mathcal{BP}^n_{n-3} \).

As mentioned in the Introduction, the case \( a = 2 \) follows from \( a = 3 \) by a general result from [19], but for completeness we reproduce the easy argument. Using double-integration as before:

\[
\rho_K(\theta) = \int_{F \in G_\theta(n, 3)} g(F) d\nu_0(F) = \int_{J \in G_\theta(n, 2)} \int_{F \in G_J(n, 3)} g(F) d\nu_J(F) d\mu_0(J),
\]

where \( \mu_0 \) and \( \nu_J \) are the Haar probability measures on the homogeneous spaces \( G_\theta(n, 2) := \{ J \in G(n, 2) | \theta \in J \} \) and \( G_J(n, 3) := \{ F \in G(n, 3) | J \subset F \} \), respectively. Denoting:

\[
h(J) = \int_{F \in G_J(n, 3)} g(F) d\nu_J(F),
\]

we see that \( h \geq 0 \) and \( \rho_K = R^*_a(h) \), implying that the body \( K_2 \) satisfying \( \rho_{K_2}^{n-2} = \rho_K \) is in \( \mathcal{BP}^n_{n-2} \).

When \( K \) is a general convex body, the result follows by approximation. It is well known (e.g. [23, Theorem 3.3.1]) that any centrally-symmetric convex body \( K \) may be approximated (for instance in the radial metric) by a series of infinitely smooth centrally-symmetric convex bodies \( \{ K^i \} \). Denoting by \( K^i_a \) the star-bodies satisfying \( \rho_{K^i_a} = \rho_{K^i_a}^{1/(n-a)} \) for \( a = 2, 3 \), we have seen that \( K^i_a \in \mathcal{BP}^n_{n-a} \). Obviously the series \( \{ K^i_a \} \) tends to \( K_a \) in the radial metric, and since \( \mathcal{BP}^n_{n-a} \) is closed under taking radial limit (see [19]), the result follows.

\[\square\]

**Remark 3.1 (Added in Proofs).** After reading a version of this note posted on the arXiv, it was communicated to us by Profs. Boris Robin and Gaoyong Zhang that Theorem 1.1 also follows from Theorems 4.3, 4.4 and 5.1 from [22]. Instead of using Koldobsky’s Lemma 3.3 which is formulated in the language of Fourier-transforms, these authors use the language of analytic families of operators to prove similar results to those of Koldobsky, which enable them to answer certain generalizations of the generalized Busemann-Petty problem.

We now turn to close a few loose ends in the proof of Corollary 1.3. Since \( \mathcal{BP}^n_k \) is closed under \( k \)-radial sums, it is immediate that if \( K^1 \) and \( K^2 \) are two convex bodies, \( L \) is their radial sum, and \( \rho_T = \rho_T^{1/(n-a)} \) for \( T = K_1, K_2, L \), then:

\[
\rho_{L}^{n-a} = \rho_L = \rho_{K_1} + \rho_{K_2} = \rho_{K^1_a}^{n-a} + \rho_{K^2_a}^{n-a},
\]

and therefore \( L \in \mathcal{BP}^n_{n-a} \). This argument of course extends to any finite radial sum of convex bodies, and since \( \mathcal{BP}^n_k \) is closed under taking limit in the radial metric, the argument extends to the entire class \( \mathcal{RC}^n \) defined in the Introduction.

It remains to prove Corollary 1.2.

**Proof of Corollary 1.2.** By Theorem 1.1, it is enough to show that for a small enough \( |\varepsilon| \) (which depends on \( n \) and \( M \)), the star-bodies \( L_{\varepsilon}^a \) defined by \( \rho_{L_{\varepsilon}^a} = \rho_{K_{\varepsilon}^{n-a}} \) are in fact convex. Since \( \rho_{L_{\varepsilon}^a} = (1 + \varepsilon \varphi)^{n-a} \), it is clear that for every \( \theta \in S^{n-1} \):

\[
|\rho_{L_{\varepsilon}^a}(\theta)| \leq f_0(\varepsilon, n), |(\rho_{L_{\varepsilon}^a})_i(\theta)| \leq f_1(\varepsilon, n, M), |(\rho_{L_{\varepsilon}^a})_{i,j}(\theta)| \leq f_2(\varepsilon, n, M),
\]

for every \( i, j = 1, \ldots, n-1 \), where \( f_0 \) tends to 1 and \( f_1, f_2 \) tend to 0, as \( \varepsilon \to 0 \). It should be intuitively clear that the convexity of \( L_{\varepsilon}^a \) depends only on the behaviour of the derivatives of
order 0, 1, and 2 of $\rho_{L_a}$, and since we have uniform convergence of these derivatives to those of the Euclidean ball as $\varepsilon$ tends to 0, $L_{a}^\varepsilon$ is convex for small enough $\varepsilon$. To make this argument formal, we follow [5], and use a formula for the Gaussian curvature of a star-body $L$ whole radial function $\rho_L$ is twice continuously differentiable, which was explicitly calculated in [20, 2.5]. In particular, it follows that $M_L(\theta)$, the Gaussian curvature of $\partial L$ (the hypersurface given by the boundary of $L$) at $\rho_L(\theta)\theta$, is a continuous function of the derivatives of order 0, 1 and 2 of $\rho_L$ at the point $\theta$. Since the Gaussian curvature of the boundary of the Euclidean ball is a constant 1, it follows that for small enough $\varepsilon$, the boundary of $L_a^\varepsilon$ has everywhere positive Gaussian curvature. By a standard result in differential geometry (e.g. [12, p. 41]), this implies that $L_a^\varepsilon$ is convex. This concludes the proof.

□

REFERENCES
8. A. A. Giannopoulos, A note on a problem of h. busemann and c.m. petty concerning sections of symmetric convex bodies, Mathematika 37 (1990), 239–244.
16. D. G. Larman and C. A. Rogers, The existence of a centrally symmetric convex body with central sections that are unexpectedly small, Mathematika 22 (1975), 164–175.


E-mail address: emanuel.milman@weizmann.ac.il

Department of Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel.
CHAPTER 4

DUAL MIXED VOLUMES AND THE SLICING PROBLEM

EMANUEL MILMAN

Advances in Mathematics 207 (2), 566-598, 2006

Abstract. We develop a technique using dual mixed-volumes to study the isotropic cons-
tants of some classes of spaces. In particular, we recover, strengthen and generalize results
of Ball and Junge concerning the isotropic constants of subspaces and quotients of $L^p$ and
related spaces. An extension of these results to negative values of $p$ is also obtained, using
generalized intersection-bodies. In particular, we show that the isotropic constant of a
convex body which is contained in an intersection-body is bounded (up to a constant) by
the ratio between the latter’s mean-radius and the former’s volume-radius. We also show
how type or cotype 2 may be used to easily prove inequalities on any isotropic measure.

1. Introduction

The main purpose of this note is to provide new types of bounds on a convex body’s
isotropic constant, by means of dual mixed-volumes with different families of bodies.

A centrally symmetric convex body $K$ in $\mathbb{R}^n$ is said to be in isotropic position if $\int_K \langle x, \theta \rangle^2 \, dx$
is constant for all $\theta \in S^{n-1}$, the Euclidean unit sphere. If in addition $K$ is of volume 1,
then its isotropic constant is defined to be the $L_K$ satisfying $\int_K \langle x, \theta \rangle^2 \, dx = L_K^2$ for all
$\theta \in S^{n-1}$. It is easy to see that every body may be brought to isotropic position using
an affine transformation, and that the isotropic position is unique modulo rotations and
homothety ([34]). Hence, for a general centrally symmetric convex body $K$ we shall denote
by $L_K$ the isotropic constant of $K$ in its isotropic position of volume 1.

A famous problem, commonly known as the Slicing Problem, asks whether $L_K$ is bounded
from above by a universal constant independent of $n$, for all centrally symmetric convex
bodies $K$ in $\mathbb{R}^n$. This was first posed in an equivalent form by J. Bourgain, who asked
whether every centrally symmetric convex body of volume 1, has an $n - 1$ dimensional
section whose volume is bounded from below by some universal constant. This is known
to be true for several families of bodies, such as sections of $L_1$, projection bodies and 1-
unconditional bodies (see [34],[3] or below). The best general bound is due to Bourgain,
who showed in [7] that $L_K \leq C n^{1/4} \log(1 + n)$. Recently, the general problem has been
reduced to the case that $K$ has finite volume-ratio ([9]).

The main idea of this note is to compare a general convex body $K$ (or its polar) with a less
general body $L$ chosen from a specific family, and thus gain some knowledge on its isotropic
constant. We shall consider two main families: unit-balls of $n$-dimensional subspaces of $L_p$.
denoted $SL_p^n$, and $k$-Busemann-Petty bodies, denoted $B^k \mathcal{P}_p^n$, which are a generalization of intersection bodies (the class $B^k \mathcal{P}_p^n$) introduced by Zhang in [40] (there they are referred to as "generalized $(n-k)$-intersection bodies", see Section 2 for definitions). The body $L$ may not be necessarily convex, but we will assume that it is a centrally symmetric star-body, defined by a continuous radial function $\rho_L(\theta) = \max \{ r \geq 0 | r \theta \in L \}$ for $\theta \in S^{n-1}$. Our main tool for comparing two star-bodies will be the dual mixed-volume of order $p$, defined in Section 2, which was first introduced by Lutwak in [29].

We will require a few more notations. Let $|x|$ denote the standard Euclidean norm of $x \in \mathbb{R}^n$, let $D_n$ denote the Euclidean unit ball and let $\sigma$ denote the Haar probability measure on $S^{n-1}$. Let $SL(n)$ denote the group of volume preserving linear transformations in $\mathbb{R}^n$, and let $\text{Vol}(B)$ denote the Lebesgue measure of the set $B \subseteq \mathbb{R}^n$ in its affine hull. Let $K^\circ$ denote the polar body to a convex body $K$.

An equivalent characterization of the isotropic position ([34]) states that it is the position which minimizes the expression $\int_K |x|^2 \, dx$, in which case the latter is equal to $nL^2_K$ if $\text{Vol}(K) = 1$. By comparing with the value of this expression in a position for which the circumradius $a(K)$ of $K$ is minimal, we immediately get the bound $L_K \leq a(K)/\sqrt{n}$. Equivalently, making this invariant to change of position or normalization, we get the following well known elementary bound on $L_K$ in terms of the outer volume-ratio of $K$:

$$L_K \leq C \inf \left\{ \left( \frac{\text{Vol}(E)}{\text{Vol}(K)} \right)^{1/n} \bigg| K \subset E , E \in SL_2^n \right\} ,$$

where $SL_2^n$ is just the class of all ellipsoids in $\mathbb{R}^n$. This was generalized in [3] by K. Ball as follows:

**Theorem (Ball).**

$$L_K \leq C \inf \left\{ \left( \frac{\text{Vol}(L)}{\text{Vol}(K)} \right)^{1/n} \bigg| K \subset L , L \in SL^n \right\} .$$

In fact, Ball showed that the expression on the right is equivalent (up to universal constants) to the so-called weak right-hand Gordon-Lewis constant $wrgl_2(X_K^*)$ of the Banach space $X_K^*$ whose unit ball is the polar of $K$. Ball showed that $wrgl_2(X^*)$ is majorized (up to a constant) by $gl_2(X)$, the Gordon-Lewis constant of $X$, and hence $L_K$ is bounded for spaces $X_K$ with uniformly bounded $gl_2$ constants. These include subspaces of $L_p$ for $1 \leq p \leq 2$, quotients of $L_q$ for $2 \leq q \leq \infty$, and spaces with a 1-unconditional basis (the latter were first shown to have a bounded isotropic constant by Bourgain). A complementary result was obtained in [21] by Junge, who showed the following (this is not explicit in his formulation but follows from the proof):

**Theorem (Junge).**

$$L_K \leq C \inf \left\{ \sqrt{p} q \left( \frac{\text{Vol}(L)}{\text{Vol}(K)} \right)^{1/n} \bigg| K \subset L , L \in SQL_p^n , 1 < p < \infty , 1/p + 1/q = 1 \right\} ,$$

where $SQL_p^n$ is the class of all unit-balls of $n$-dimensional subspaces of quotients of $L_p$, and $q = p^*$ is the conjugate exponent to $p$. In fact, Junge showed that $L_p$ may be replaced by any Banach space $X$ with bounded $gl_2(X)$ such that $X$ has finite type, in which case $\sqrt{p} q$ above should be replaced by some constant depending on $X$. ```
As evident from their more general formulations, the results of Ball and Junge described above make heavy use of non-trivial Functional Analysis and Operator Theory, and as a result the geometric intuition behind the Slicing Problem is substantially lost. Of course, this is to be expected if the conditions on the space $X_K$ are formulated using Operator Theory notions, such as (variants of) the Gordon-Lewis property. But for classical spaces such as subspaces or quotients of $L_p$, one may hope to simplify the approach, derive better bounds on $L_K$, and unify Ball and Junge’s results into a single framework. Using an elementary argument, geometric in nature, we show the following generalizations of (1.1) and partial strengthening of (1.2) (the term “partial” refers to the fact that we restrict $L$ to the class $SL_p^n$ or $QL_q^n$ defined below), for a convex isotropic body $K$ with Vol $(K) = Vol (D_n)$:

**Theorem 1.1.**

$$L_K \leq C \inf \left\{ \frac{\sqrt{p_0}}{M_p(L)} \mid K \subset L, L \in SL_p^n, p \geq 0 \right\},$$

where $p_0 = \max(1, \min(p, n))$, $M_p(L) = \left( \int_{S^{n-1}} \|x\|_L^p \, d\sigma(x) \right)^{1/p}$ for $p > 0$, and by passing to the limit, $M_0(L) = \exp \left( \int_{S^{n-1}} \log \|x\|_L \, d\sigma(x) \right)$.

**Theorem 1.1 (').**

$$L_K \leq CT_2(X_K) \frac{M_2(K)}{M_2(K)},$$

where $T_2(X_K)$ is the (Gaussian) type-2 constant of $X_K$.

**Theorem 1.2.**

$$L_K \leq C \inf \left\{ \mathcal{L}_k \tilde{M}_k(L) \mid K \subset L, L \in BP_k^n, k = 1, \ldots, n-1 \right\},$$

where $\mathcal{L}_k$ denotes the maximal isotropic constant of centrally symmetric convex bodies in $\mathbb{R}^k$ and $\tilde{M}_k(L) = \left( \int_{S^{n-1}} \rho_L(x)^k \, d\sigma(x) \right)^{1/k}$. We emphasize again that $BP_1^n$ is exactly the class of intersection bodies.

Indeed, these are all generalizations of (1.1) and (1.2), since by passing to polar coordinates and applying Jensen’s inequality (for $p, k > 0$):

$$(1.3) \frac{1}{M_p(L)} \leq \tilde{M}_k(L) \leq \left( \frac{\text{Vol} (L)}{\text{Vol} (D_n)} \right)^{1/n},$$

and since $T_2(X_K) \leq C \sqrt{p}$ by Kahane’s inequality when $K \in SL_p^n$ for $p \geq 2$. This also applies to Theorem 2, since any $K \in SL_p^n$ for $0 < p \leq 2$ (and in particular $p = 1$) is an intersection body (see [24]), and hence a $k$-Busemann-Petty body for all $k \geq 1$ ([19],[33]).

We also have the following dual counterparts to Theorems 1.1 and 1.2, for a convex isotropic body $K$ with Vol $(K) = Vol (D_n)$:

**Theorem 1.3.**

$$L_K \leq C \inf \left\{ \sqrt{p_0} M_p^*(T(L)) \mid K \subset L, L \in QL_q^n, T \in SL(n), 1 \leq q \leq \infty, 1/p + 1/q = 1 \right\},$$

where $QL_q^n$ is the class of all unit-balls of $n$-dimensional quotients of $L_q$, $p_0$ is defined as above for $p = q^*$, and $M_p^*(G) = M_p(G^o)$. 
This is indeed a (partial) strengthening of (1.2), since by Lemma 4.5 (see also the Mean Norm Corollary below), there exists a position $T \in SL(n)$ of $L \in QL_q^n$ such that:

$$M_p^u(T(L)) \leq C \sqrt{\rho_0} \left( \frac{\text{Vol}(L)}{\text{Vol}(D_n)} \right)^{1/n}.$$ 

It is also interesting to note that the proof of Theorem 1.3, although derived independently, closely resembles Bourgain’s proof that $L_K \leq Cn^{1/4}\log(1+n)$.

**Theorem 1.4.**

$$L_K \leq C \inf \left\{ \frac{\mathcal{L}_{2k}^2}{M_k(T(L))} \mid L \subset K^\circ, L \in \mathcal{B}_p^n, T \in SL(n), k = 1, \ldots, \lfloor n/3 \rfloor \right\}.$$ 

Using an analogue of Lemma 4.5 (stated in the Mean Radius Corollary below), we may deduce the following bound on $L_K$ for polars of bodies in $C\mathcal{B}_p^n$, the class of convex $k$-Busemann-Petty bodies:

$$L_K \leq C \inf \left\{ \frac{\mathcal{L}_{2k}^2}{M_k(T(L))} \left( \frac{\text{Vol}(L)}{\text{Vol}(K)} \right)^{1/n} \mid K \subset L^\circ, L \in C\mathcal{B}_p^n, k = 1, \ldots, \lfloor n/3 \rfloor \right\}.$$ 

Since Jensen’s inequality (1.3) is usually strict, it is not hard to construct examples for which Theorem 1.1 asymptotically out-performs Junge’s bound. Indeed, for $K = [-1,1]^n$, it is well known (see Section 6) that $K$ is isomorphic to a body $L \in SL_p^n$, for $p = \log n$. Junge’s bound therefore implies $L_K \leq C\sqrt{\log n}$, while Theorem 1.1 gives $L_K \leq C$, since $M_p(L) \simeq M_p(K) \simeq \sqrt{\log n} \left( \frac{\text{Vol}(K)}{\text{Vol}(D_n)} \right)^{1/n}$.

As mentioned above, Theorems 1.1 and 1.3 imply, in particular, that $L_K \leq C\sqrt{p}$ for $K \in SL_p^n$ and $p \geq 1$, and $L_K \leq q^*$ for $K \in QL_q^n$ and $q > 1$. We note that this is not contained in Junge’s result (1.2). The strength of (1.2) is that it applies simultaneously to all subspaces of quotients of $L_p$, which our method does not handle. Ironically, this is also its drawback, if one is interested in proper subspaces or quotients only: it gives the same bound on $L_K$ in either case. Therefore, one cannot hope to have a good bound for $SL_p^n$ with $1 \leq p < 2$ ($QL_q^n$ with $q > 2$) without solving the Slicing Problem, because this would imply the same bound for $QL_q^n(SL_p^n)$ in that range, which already contain all convex bodies. To fill the bound for $SL_p^n$ with $1 \leq p < 2$ ($QL_q^n$ with $q > 2$), one needs to use Ball’s result in its general form (or simply use (1.1) combined with the fact that $SL_p^n \subset SL_q^n$ for $1 \leq p \leq 2$; by duality $QL_q^n \subset QL_{\infty}^n$ for $q \geq 2$, implying that the bodies in $QL_q^n$ have finite outer volume-ratio as projection bodies). We therefore see that Theorems 1.1 and 1.3 combine the ranges $1 \leq p < 2$ and $p \geq 2$ into a single framework.

Evidently, Theorem 1.1’ has a somewhat different flavor, and indeed its proof is totally different from the proofs of the other Theorems. The proof is based on a simple yet effective framework for combining isotropic measures with type and cotype 2, which is introduced in Section 3 (this section may be read independently from the rest of this note). This framework also enables us to easily recover several known lemmas on John’s maximal volume ellipsoid position (originally proved using Operator Theory techniques), which we use in the proof of Lemma 4.5 (mentioned above). We remark that Theorem 1.1’ also follows from the work in [10] but in a more complicated manner.
The other Theorems are all proved using another technique, involving dual mixed-volumes. Theorems 1.1 and 1.3 are proved in Section 4, and Theorems 1.2 and 1.4 are proved in Section 5. In Section 6, we give several corollaries of our main Theorems, some of which are mentioned below.

Using the known fact that $L_K$ is always bounded from below, Theorems 1.1’, 1.1 and 1.2, immediately yield the following useful corollary, for an isotropic convex body $K$ with $\text{Vol}(K) = \text{Vol}(D_n)$:

**Mean Norm/Radius Corollary.**

1. $M_2(K) \leq CT_2(X_K)$.
2. If $K \in SL_n^p$ ($p > 0$), then $M_p(K) \leq C\sqrt{p}$.
3. If $K \in BP_n^k$ ($k = 1, \ldots, n-1$), then $\tilde{M}_k(K) \geq C/L_k$.

Jensen’s inequality in (1.3) shows that these bounds are tight (to within a constant) for $p, k, T_2(X_K) \leq C$. One should also keep in mind that if $K^\circ$ is in isotropic position, this corollary is applicable to $K^\circ$, providing different inequalities.

In addition, although this is a direct consequence of the extended formulation of Junge’s Theorem (and also of Theorems 1.1 and 1.3), the following corollary about a centrally symmetric convex polytope $P$ is worth explicit stating:

**Polytope Corollary.**

1. If $P$ has $2m$ facets then $L_P \leq C\sqrt{\log(1+m)}$.
2. If $P$ has $2m$ vertices then $L_P \leq C\log(1+m)$.

In particular, this implies that Gluskin’s probabilistic construction in [16] of two convex bodies $K_1$ and $K_2$ with Banach-Mazur distance of order $n$, satisfies $L_{K_1}, L_{K_2} \leq C\log(1+n)$.

Theorem 1.2 should be understood as a partial complimentary result to Theorem 1.1. The reason for this may be better explained, if we first consider a second generalization of intersection bodies, introduced by Koldobsky in [25]. We shall call these bodies $k$-intersection bodies and denote this class of bodies by $I_k^n$. It was shown in [25] that $BP_k^n \subset I_k^n$, and the question of whether $BP_k^n = I_k^n$ remains open (see [33] for an account of recent progress in this direction). The class $I_k^n$ satisfies a certain characterization of being embedded in $L_p$, which has been continued analytically to the negative value $p = -k$, so in some sense $I_k^n = SL_{-k}^n$. Therefore, in some sense, $BP_k^n \subset SL_{-k}^n$, hence our initial remark.

The class of star-bodies $BP_k^n$ seems at first glance a non-natural object to work with when studying convex bodies. Nevertheless, we describe in Section 6 several potential ways in which this object may be harnessed to our advantage.

**Acknowledgments.** I would like to deeply thank my supervisor Prof. Gideon Schechtman for many informative discussions, and especially for believing in me and allowing me to pursue my interests. I would also like to thank the referee for many helpful remarks.

2. **Definitions and Notations**

A convex body $K$ will always refer to a compact, convex set in $\mathbb{R}^n$ with non-empty interior. We will always assume that the bodies in question are centrally symmetric, i.e. $K = -K$. The equivalence between convex bodies and norms in $\mathbb{R}^n$ is well known, with the
Passing to the limit as $p$ denoted by $\tilde{\nu}$ of dual mixed-volumes (see [31]): it is obvious that $\tilde{\nu}$ correspondence $\|x\|_K = \min \{ t > 0 | x/t \in K \}$. The associated normed space $(\mathbb{R}^n, \|\cdot\|_K)$ will be denoted by $X_K$. The dual norm is defined as $\|x\|_K^* = \sup_{y \in K} \langle x, y \rangle$, and its associated unit-ball is called the polar body to $K$, and denoted $K^\circ$. The dual normed space $(\mathbb{R}^n, \|\cdot\|_K^*)$ is denoted by $X_K^*$ ($= X_{K^\circ}$). We will say that a convex-body $K$ is 1-unconditional, or simply unconditional, with respect to the given Euclidean structure (which we always assume to be fixed), if $(x_1, \ldots, x_n) \in K$ implies $(\pm x_1, \ldots, \pm x_n) \in K$ for all possible sign assignments.

We will also work with general star-bodies $L$, which are star-shaped bodies, meaning that $tL \subset L$ for all $t \in [0, 1]$, with the additional requirement that their radial function $\rho_L$ is a continuous function on $S^{n-1}$. The radius of $L$ in direction $\theta \in S^{n-1}$ is defined as $\rho_L(\theta) = \max \{ r \geq 0 | r\theta \in L \}$. For a general star-body $L$, we define its Minkowski functional $\|x\|_L$ in the same manner as for a convex body (so $\|x\|_L$ is no longer necessarily a norm). Obviously, $\rho_L(\theta) = 1/\|\theta\|_L$ for all $\theta \in S^{n-1}$.

By identifying between a star-body and its radial function, a natural metric arises on the space of star-bodies. The radial metric, denoted by $d_r$, is defined as:

$$d_r(L_1, L_2) = \sup_{\theta \in S^{n-1}} |\rho_{L_1}(\theta) - \rho_{L_2}(\theta)|.$$

As mentioned in the Introduction, our main tool for comparing two star-bodies $L_1$ and $L_2$ will be the dual mixed-volume of order $p \in \mathbb{R}$, introduced by Lutwak in [29] (see also [31]), and defined as:

$$\tilde{V}_p(L_1, L_2) = \frac{1}{n} \int_{S^{n-1}} \rho_{L_1}(x)^p \rho_{L_2}(x)^{n-p} \, dx$$

(note that the integration is w.r.t. the Lebesgue measure on $S^{n-1}$). By polar integration, it is obvious that $\tilde{V}_p(L, L) = \text{Vol}(L)$ for all $p$. We will also use the following useful property of dual mixed-volumes (see [31]):

$$\tilde{V}_p(T(L_1), T(L_2)) = \tilde{V}_p(L_1, L_2),$$

for any $T \in SL(n)$ and $p \in \mathbb{R}$. We also constantly use the well known formula for the volume of the Euclidean unit ball $D_n$:

$$\text{Vol}(D_n) = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)}.$$

Several useful notations for a star-body $L$ will be used. For $p > 0$, the $p$-th mean-norm, denoted by $M_p(L)$, is defined as:

$$M_p(L) = \left( \int_{S^{n-1}} \|x\|_L^p \, d\sigma(x) \right)^{1/p}.$$

Passing to the limit as $p \to 0$, we define $M_0(L) = \exp \left( \int_{S^{n-1}} \log \|x\|_L \, d\sigma(x) \right)$. We will define the mean-norm as $M(L) = M_1(L)$. The $p$-th mean-width, denoted $M_p^*(L)$, is defined as $M_p^*(L) = M_p(L^*)$, and as usual, the mean-width is defined as $M^*(L) = M_1^*(L)$. The $p$-th mean-radius, denoted by $\tilde{M}_p(L)$, is defined as:

$$\tilde{M}_p(L) = \left( \int_{S^{n-1}} \rho_L(x)^p d\sigma(x) \right)^{1/p}.$$
We will define the mean-radius as \( \overline{M}(L) = \overline{M}_1(L) \). The minimal \( a, b > 0 \) for which \( 1/a \|x\|_L \leq b \|x\| \) \( \forall x \in X \), will be denoted by \( a(L) \) and \( b(L) \), respectively. Geometrically, \( a(L) \) and \( 1/b(L) \) are the radii of the circumscribing and inscribed Euclidean balls of \( L \), respectively. The expression \( (\text{Vol}(L)/\text{Vol}(D_n))^{1/n} \) will be referred to as the volume-radius of \( L \). The infimum of \( (\text{Vol}(L)/\text{Vol}(\mathcal{E}))^{1/n} \) over all ellipsoids \( \mathcal{E} \) contained in \( L \) is called the volume-ratio of \( L \). Similarly, the infimum of \( (\text{Vol}(\mathcal{E})/\text{Vol}(L))^{1/n} \) over all ellipsoids \( \mathcal{E} \) containing \( L \) is called the outer volume-ratio of \( L \). A position of a body \( L \) is a volume preserving linear image of \( L \), i.e. \( T(L) \) for \( T \in SL(n) \).

Going back to convex bodies and normed spaces, we now define the (Gaussian) type and cotype 2 constants of a normed space \( X \), for any \( m \). We denote \( \rho \) \((\text{31}), (\text{14})\) to \( m \) for any \( \rho \) \( \geq 1 \) and \( \rho \) \( \geq 1 \) for \( \rho \) \( \geq 1 \) and any \( x_1, \ldots, x_m \in X \), where \( g_1, \ldots, g_m \) are independent real-valued standard Gaussian r.v.'s on a common probability space \( (\Omega, d\omega) \). Similarly, the (Gaussian) cotype-2 constant of \( X \), denoted \( C_2(X) \), is the minimal \( C > 0 \) for which:

\[
\left( \int_{\Omega} \| \sum_{i=1}^{m} g_i(\omega)x_i \|^2 d\omega \right)^{1/2} \leq T \left( \sum_{i=1}^{m} \|x_i\|^2 \right)^{1/2}
\]

for any \( m \geq 1 \) and any \( x_1, \ldots, x_m \in X \). We will not distinguish between the Gaussian and the Rademacher type (cotype) 2 constants, since it is well known that the former constant is always majorated by the latter one (e.g. \([\text{36}]\)), and all our results will involve upper bounds in terms of the Gaussian type (cotype) 2.

We will often identify between a normed space and its unit-ball. In particular, for the infinite dimensional Banach space \( L_p = L_p([0,1], dx) \), whenever the expression ”sections of \( L_p \)” is used, we will mean sections of its unit-ball. And when the expression ”quotients of \( L_p \)” is used, we might refer to the unit-balls of these quotient spaces.

Throughout the paper, all constants used will be universal, independent of all other parameters, and in particular, independent of \( n \). We reserve \( C, C', C_1, C_2 \) to denote these constants, which may take different values on separate instances. We will write \( A \simeq B \) to signify that \( C_1 A \leq B \leq C_2 A \) with universal constants \( C_1, C_2 > 0 \).

For the results of Sections 5 and 6, we shall need to define the class of \( k \)-Busemann-Petty bodies, introduced by Zhang in \([\text{40}]\) (there they are referred to as ”generalized \((n-k)\)-intersection bodies”). These bodies represent a generalization of the notion of an intersection body. For completeness, we give the appropriate definitions below.

**Definition.** A star body \( K \) is said to be an intersection body of a star body \( L \), if \( \rho_K(\theta) = \text{Vol}(L \cap \theta^\perp) \) for every \( \theta \in S^{n-1} \). \( K \) is said to be an intersection body, if it is the limit in the radial metric \( d_r \) of intersection bodies \( \{K_i\} \) of star bodies \( \{L_i\} \). This is equivalent (e.g. \([\text{31}], [\text{14}]\)) to \( \rho_K = \mathcal{R}^*(d\mu) \), where \( \mu \) is a non-negative Borel measure on \( S^{n-1} \), \( \mathcal{R}^* \) is the dual transform (as in \((2.3)\)) to the Spherical Radon Transform \( R : C(S^{n-1}) \to C(S^{n-1}) \), which
is defined for \( f \in C(S^{n-1}) \) as:

\[
R(f)(\theta) = \int_{S^{n-1} \cap \theta^\perp} f(\xi)d\sigma_{n-1}(\xi),
\]

where \( \sigma_{n-1} \) the Haar probability measure on \( S^{n-2} \) (and we have identified \( S^{n-2} \) with \( S^{n-1} \cap \theta^\perp \)).

Let \( G(n,m) \) denote the Grassmann manifold of all \( m \)-dimensional linear subspaces of \( \mathbb{R}^n \). Generalizing the Spherical Radon Transform is the \( m \)-dimensional Spherical Radon Transform \( R_m \), acting on spaces of continuous functions as follows:

\[
R_m : C(S^{n-1}) \longrightarrow C(G(n,m)) \nonumber
\]

\[
R_m(f)(E) = \int_{S^{n-1} \cap E} f(\theta)d\sigma_m(\theta),
\]

where \( \sigma_m \) is the Haar probability measure on \( S^{m-1} \) (and we have identified \( S^{m-1} \) with \( S^{n-1} \cap E \)). Notice that for a star-body \( L \) in \( \mathbb{R}^n \):

\[
R_m(\rho_k^m)(E) = \frac{\text{Vol}(L \cap E)}{\text{Vol}(D_m)} \quad \forall E \in G(n,m).
\]

The dual transform is defined on spaces of \textit{signed} Borel measures \( \mathcal{M} \) by:

\[
R^*_m : \mathcal{M}(G(n,m)) \longrightarrow \mathcal{M}(S^{n-1}) \nonumber
\]

\[
\int_{S^{n-1}} f R^*_m(d\mu) = \int_{G(n,m)} R_m(f)d\mu \quad \forall f \in C(S^{n-1}),
\]

and for a measure \( \mu \) with continuous density \( g \), the transform may be explicitly written in terms of \( g \) (see [40]):

\[
R^*_m g(\theta) = \int_{\theta \in E \in G(n,m)} g(E)d\nu_m(E),
\]

where \( \nu_m \) is the Haar probability measure on \( G(n-1, m-1) \).

**Definition.** A star body \( K \) is said to be a \( k \)-Busemann-Petty body if \( \rho_k^m \equiv R^*_m(d\mu) \), where \( \mu \) is a non-negative Borel measure on \( G(n, n-k) \). We shall denote the class of such bodies by \( \mathcal{BP}_k^n \).

Choosing \( k = 1 \), for which \( G(n, n-1) \) is isometric to \( S^{n-1}/\mathbb{Z}_2 \) by mapping \( H \) to \( S^{n-1} \cap H^\perp \), and noticing that \( R \) is equivalent to \( R_{n-1} \) under this map, we see that \( \mathcal{BP}_1^n \) is exactly the class of intersection bodies.

To conclude this section, we mention that we always work with the radial metric topology on the space of star-bodies. Equivalently, we always work with the maximum norm on the space of continuous functions on \( S^{n-1} \). So whenever an expression of the following form appears:

\[
f = \int f_\alpha d\mu(\alpha),
\]

where \( f \) and \( \{f_\alpha\} \) are continuous functions on \( S^{n-1} \), the convergence of the integral should be understood in the maximum norm.
3. COMBINING ISOTROPIC MEASURES WITH TYPE / COTYPE 2

In this section we introduce a very simple yet effective framework, which demonstrates how to utilize isotropic measures associated with a convex body $K$, to give bounds on $M_2(K)$ and $M_2^*(K)$ in terms of the type-2 and cotype-2 constants of $X_K$ and $X_K^*$. As an immediate corollary, we revive a couple of known (yet partially forgotten) lemmas on John’s maximal volume ellipsoid position, one of which will be used in Section 4 to improve the bound on the isotropic constant of quotients of $L_q$. Another immediate corollary of this framework is that $L_K$ is always bounded by $T_2(X_K)$.

Recall that a Borel measure $\mu$ on $\mathbb{R}^n$ is said to be isotropic if:

$$\int_{\mathbb{R}^n} \langle x, \theta \rangle^2 d\mu(x) = |\theta|^2 \quad \forall \theta \in \mathbb{R}^n.$$  

This is easily seen to be equivalent to:

$$\int_{\mathbb{R}^n} \langle x, \theta_1 \rangle \langle x, \theta_2 \rangle d\mu(x) = \langle \theta_1, \theta_2 \rangle \quad \forall \theta_1, \theta_2 \in \mathbb{R}^n.$$  

The main point of this section is the following easy yet useful observation:

**Lemma 3.1.** Let $v_i \in \mathbb{R}^n$ and $\lambda_i > 0$, for $i = 1, \ldots, m$, be such that $\mu = \sum_{i=1}^m \lambda_i \delta_{v_i}$ is an isotropic measure. Let $\{g_i\}_{i=1}^m$ be a sequence of independent real-valued standard Gaussian r.v.’s, and define the r.v. $\Lambda_\mu$ as:

$$\Lambda_\mu = \sum_{i=1}^m g_i \sqrt{\lambda_i} v_i.$$  

Then $\Lambda_\mu$ is an $n$-dimensional standard Gaussian.

**Proof.** Obviously $\Lambda_\mu$ is a zero mean Gaussian r.v., so it remains to show that its correlation matrix is the identity. Indeed, from the independence of the $g_i$’s and the isotropy of $\mu$:

$$E(\langle \Lambda_\mu, \theta_1 \rangle \langle \Lambda_\mu, \theta_2 \rangle) = E\left(\sum_{i,j=1}^m g_i g_j \sqrt{\lambda_i} \sqrt{\lambda_j} \langle v_i, \theta_1 \rangle \langle v_j, \theta_2 \rangle\right) =$$

$$E\left(\sum_{i=1}^m g_i^2 \lambda_i \langle v_i, \theta_1 \rangle \langle v_i, \theta_2 \rangle\right) = \sum_{i=1}^m \lambda_i \langle v_i, \theta_1 \rangle \langle v_i, \theta_2 \rangle = \langle \theta_1, \theta_2 \rangle.$$  

By taking the Fourier transform of the densities on both sides of (3.1), or by projecting them onto an arbitrary direction, we get:

$$\exp(-|x|^2) = \Pi_{i=1}^m \left(\exp\left(-\langle x, v_i \rangle^2\right)\right)^{\lambda_i}.$$  

This formulation, which is easy to check directly, has been used by many authors (e.g. [39], [2]), mostly with connection to John’s decomposition of the identity. The advantage of Lemma 3.1 is that we may work directly on the Gaussian r.v.’s and use type and cotype estimates on $\|\Lambda_\mu\|$, as summarized in the following Proposition.
Proposition 3.2. Let $K$ denote a convex body and let $\mu$ be any finite, compactly supported, isotropic measure. Then:

$$\frac{1}{C_2(X_K)} \left( \int \|x\|^2_K d\mu(x) \right)^{1/2} \leq \sqrt{n}M_2(K) \leq T_2(X_K) \left( \int \|x\|^2_K d\mu(x) \right)^{1/2}$$

Proof. First, assume that $\mu$ is a discrete isotropic measure supported on finitely many points, of the form $\mu = \sum_{i=1}^m \lambda_i \delta_{v_i}$. Then by Lemma 3.1, denoting $\{g_i\}_{i=1}^m$ and $\{g'_i\}_{i=1}^n$ two sequences of independent standard Gaussian r.v.’s on a common probability space $(\Omega, d\omega)$, we have:

$$\int_{\Omega} \| \sum_{i=1}^m g_i(\omega) \sqrt{\lambda_i} v_i \|_K^2 d\omega = \int_{\Omega} \| \sum_{i=1}^n g'_i(\omega) e_i \|_K^2 d\omega = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \|x\|_K^2 e^{-|x|^2/2} dx = \frac{\int_{0}^{\infty} e^{-r^2/2} r^{n+1} dr}{(2\pi)^{n/2}} \int_{S^{n-1}} \|\theta\|_K^2 d\theta = nM_2(K)^2,$$

where the last equality is a standard calculation (e.g. [36]). But on the other hand, using the type-2 condition on $X_K$, we see that the initial expression on the left is bounded from above by:

$$T_2(X_K)^2 \sum_{i=1}^m \| \sqrt{\lambda_i} v_i \|_K^2 = T_2(X_K)^2 \int \|x\|_K^2 d\mu(x).$$

Taking square root, the type-2 upper bound follows for a discrete measure $\mu$, and the cotype-2 lower bound follows similarly.

When $\mu$ is a general isotropic measure, we approximate $\mu$ by a series of discrete (not necessarily isotropic) measures $\mu_{\varepsilon} = \sum_{i=1}^{m_{\varepsilon}} \lambda_{i}^{\varepsilon} \delta_{v_{i}^{\varepsilon}}$, where $\varepsilon > 0$ is a parameter which will tend to 0. Since the set of discrete finitely supported measures is dense in the space of compactly supported Borel measures on $\mathbb{R}^n$ in the $\mathcal{w}'$-topology, we may choose $\mu_{\varepsilon}$ so that as linear functionals, the values of $\mu$ and $\mu_{\varepsilon}$ on the following $n(n+1)/2 + 1$ continuous functions are $\varepsilon$ close:

$$\left| \int x_i x_j d\mu_{\varepsilon}(x) - \delta_{i,j} \right| = \left| \int x_i x_j d\mu_{\varepsilon}(x) - \int x_i x_j d\mu(x) \right| < \varepsilon,$$

for all $1 \leq i \leq j \leq n$ and:

$$\left(3.2\right) \left| \int \|x\|_K^2 d\mu_{\varepsilon}(x) - \int \|x\|_K^2 d\mu(x) \right| < \varepsilon.$$

We see that $\mu_{\varepsilon}$ is chosen to be almost isotropic, but we do not know how to guarantee this in general. Now, repeating the proof of Lemma 3.1, we see that $\Lambda_{\mu_{\varepsilon}}$ in (3.1) is a Gaussian r.v. whose correlation matrix is almost the identity (up to an $l_\infty$ error of $\varepsilon$ w.r.t. the standard basis). Therefore sending $\varepsilon$ to 0, $\Lambda_{\mu_{\varepsilon}}$ tends to an $n$-dimensional standard Gaussian r.v. almost surely, implying that $\int \| \sum_{i=1}^{m_{\varepsilon}} g_i(\omega) \sqrt{\lambda_i} v_i \|_K^2 d\omega$ tends to $\int \| \sum_{i=1}^{n} g'_i(\omega) e_i \|_K^2 d\omega = nM_2(K)^2$. Since by the discrete case:

$$\int \| \sum_{i=1}^{m_{\varepsilon}} g_i(\omega) \sqrt{\lambda_i} v_i \|_K^2 d\omega \leq T_2(X_K)^2 \int \|x\|_K^2 d\mu_{\varepsilon}(x),$$

and $\int \|x\|_K^2 d\mu_{\varepsilon}(x)$ tends to $\int \|x\|_K^2 d\mu(x)$ by (3.2), this completes the proof. $\square$
One of the most useful isotropic measures associated to the geometry of a convex body $K$, comes from John’s decomposition of the identity, when $K$ is put in John’s maximal volume ellipsoid position: if $D_n$ is the ellipsoid of maximal volume inside $K$, there exist contact points $\{v_i\}$ of $D_n$ and $K$ and positive scalars $\{\lambda_i\}$, such that $\mu_K = \sum_{i=1}^{m} \lambda_i \delta_{v_i}$ is isotropic. Since $|v_i| = 1$, it immediately follows that $\sum_{i=1}^{m} \lambda_i = n$. Applying Proposition 3.2 with the measure $\mu_K$, first with $K$ and then with $K^0$, we immediately have as a corollary the following two known inequalities. The first essentially appears in [32], and in [36] with a worse constant, and the second appears in [13]. Both in [13] and in [32], the proofs rely on Operator Theory, whereas in our approach the elementary geometric flavor is retained, and both proofs are unified into a single framework.

**Corollary 3.3.** Let $K$ be a convex body in John’s maximal volume ellipsoid position. Then:

$$M_2(K)/b(K) \geq 1/C_2(X_K),$$

$$M_2^*(K)b(K) \leq T_2(X_K^*).$$

**Proof.** The $b(K)$ terms are simply normalizations to the case that $D_n$ is indeed the ellipsoid of maximal volume inside $K$. It remains to notice that $|v_i| = \|v_i\|_K = \|v_i\|_{K^*} = 1$, as contact points between $D_n$ and $K$. Since $\sum_{i=1}^{m} \lambda_i = n$, we have:

$$\left(\sum_{i=1}^{m} \lambda_i (\|v_i\|_K)^2\right)^{1/2} = \left(\sum_{i=1}^{m} \lambda_i (\|v_i\|_{K^*})^2\right)^{1/2} = \sqrt{n}.$$

The assertions now clearly follow from Proposition 3.2. \qed

**Remark 3.1.** The other two inequalities:

$$M_2(K)/b(K) \leq T_2(X_K),$$

$$M_2^*(K)b(K) \geq 1/C_2(X_K^*),$$

are trivial and loose. The first follows from $M_2(K) \leq b(K)$, and the second from Urysohn’s inequality:

$$M_2^*(K) \geq M^*(K) \geq \left(\frac{\text{Vol}(K)}{\text{Vol}(D_n)}\right)^{1/n} \geq \frac{1}{b(K)}.$$

By duality, we have:

**Corollary 3.4.** Let $K$ be a convex body in Lowner’s minimal volume outer ellipsoid position. Then:

$$M_2^*(K)/a(K) \geq 1/C_2(X_K^*),$$

$$M_2(K)a(K) \leq T_2(X_K).$$

Corollary 3.4 shows that having type-2 implies having finite outer volume-ratio (this will be evident in the proof of the next Theorem), so it is not surprising that we get the following useful bound on the isotropic constant, when placing the body in Lowner’s outer ellipsoid position. What is a little more surprising, is that we manage to get the same bound by putting the body in the isotropic position, and directly applying Proposition 3.2 on the (properly normalized) uniform measure on $K$. The latter part may also be shown to follow from Theorem 1.4 in [10].
Theorem 3.5. Let $K$ be a convex body. Then:

\[(3.3) \quad L_K \leq C \inf \left\{ T_2(X_L) \left( \frac{\text{Vol}(L)}{\text{Vol}(K)} \right)^{1/n} \right\} \] \[\quad \text{if } K \subset L \text{ is a convex body.} \]

In addition, if $\text{Vol}(K) = 1$ and $K$ is in Lowner’s minimal volume outer ellipsoid position or in isotropic position, then:

\[L_K \leq C \frac{T_2(X_K)}{\sqrt{n}M_2(K)}.\]

Proof. Since (3.3) is invariant under homothety, we may assume that $\text{Vol}(K) = 1$. Now let $L$ be any convex body containing $K$, and assume that $T(L)$ is in Lowner’s minimal volume outer ellipsoid position, where $T \in SL(n)$. By Corollary 3.4 and Jensen’s inequality (as in (1.3)):

\[a(T(L)) \leq \frac{T_2(X_L)}{M_2(T(L))} \leq C \sqrt{n} \left( \frac{\text{Vol}(L)}{\text{Vol}(K)} \right)^{1/n} T_2(X_L).\]

Using the characterization of $L_K$ mentioned in the Introduction, we immediately have:

\[L_K^2 \leq \frac{1}{n} \int_{T(K)} |x|^2 dx \leq \frac{1}{n} a(T(L))^2 \leq \left( C \left( \frac{\text{Vol}(L)}{\text{Vol}(K)} \right)^{1/n} T_2(X_L) \right)^2.\]

Evidently, the above argument also proves the second part of the Theorem when $K$ is in Lowner’s minimal volume outer ellipsoid position. When $K$ is in isotropic position, we apply Proposition 3.2 to the isotropic measure $d\mu = 1/L_K^2 \chi_K dx$, yielding:

\[\sqrt{n}M_2(K) \leq T_2(X_K) \left( \frac{1}{L_K} \left( \int_K \|x\|^2_K \right)^{1/2} \right)^2 \leq T_2(X_K)/L_K.\]

The assertion therefore follows (even without a constant). 

Remark 3.2. For completeness, it is worthwhile to mention that a different form of Theorem 3.5 may be derived from a deeper result of Milman and Pisier, who showed in [35] that the volume-ratio of $K$ is bounded from above by $CC_2(X_K) \log C_2(X_K)$ (this is an improvement over the initial bound showed in [11]). Using another deep result, the reverse Blaschke-Santalo inequality ([11], see (4.11)), this implies that the outer volume-ratio of $K$ is bounded from above by $C' C_2(X_K^*) \log C_2(X_K^*)$, so the same argument as above gives:

\[L_K \leq C \inf \left\{ C_2(X_L^*) \log C_2(X_L^*) \left( \frac{\text{Vol}(L)}{\text{Vol}(K)} \right)^{1/n} \right\} \] \[\quad \text{if } K \subset L \text{ is a convex body.} \]

Since $C_2(X_L^*) \leq T_2(X_L) \leq C_2(X_L^*) \|\text{Rad}(X_L)\|$, we see that the two forms are very similar, but elementary examples show that neither form out-performs the other.

Since it is well known (e.g. [36]) that subspaces of $L_p$, for $p \geq 2$, have a type-2 constant of the order of $\sqrt{p}$ (this is a consequence of Kahane’s inequality), we immediately have the following Corollary of Theorem 3.5.
Corollary 3.6.

\[ L_K \leq C \inf \left\{ \sqrt[p]{\left( \frac{\text{Vol}(L)}{\text{Vol}(K)} \right)}^{1/n} \middle| K \subset L, \ L \in SL_n^p, \ p \geq 2 \right\} . \]

We conclude this section by giving another application of Proposition 3.2. In principle, it seems useful to apply it on any isotropic measure which is naturally associated to a convex body in certain special positions. Fortunately, in [15], Giannopoulos and Milman have derived a framework to generate such measures, by considering bodies in minimum quermassintegral positions. We will only give the following application for the minimal surface-area position, i.e. the position for which \( \text{Vol}(\partial T(K)) \) is minimal for all \( T \in SL(n) \), which was characterized by Petty in [38]. Recall that \( \sigma_K \), the area measure of \( K \) is defined on \( S^{n-1} \) as:

\[ \sigma_K(A) = \nu(\{ x \in \partial K | n_K(x) \in A \}) , \]

where \( n_K(x) \) denotes an outer normal to \( K \) at \( x \) and \( \nu \) is the \( n-1 \) dimensional surface measure on \( K \).

Proposition 3.7. Let \( K \) be a convex body in minimal surface-area position. Then:

\[ \frac{1}{C_2(X_K)} \leq \frac{M_2(K)}{\left( \frac{1}{\text{Vol}(\partial K)} \int_{S^{n-1}} \| x \|^2 d\sigma_K(x) \right)^{1/2}} \leq T_2(X_K). \]

Proof. It was shown in [38] that \( K \) is in minimal surface-area position iff \( n/\text{Vol}(\partial K) d\sigma_K \) is isotropic. Applying Proposition 3.2 with \( \sigma_K \) yields the claimed inequalities. \( \square \)

4. Sections and Quotients of \( L_p \)

As seen in the previous section, it is actually pretty straightforward to obtain a bound on the isotropic constant of any convex body \( K \) for which we have control over \( T_2(X_K) \), since in that case \( K \) has bounded outer volume-ratio. In particular, this applies for sections of \( L_p \), at least for \( p \geq 2 \). In this section, we introduce a new technique involving dual mixed-volumes, which is well adapted to deal specifically with integral representations of \( \| \cdot \|_t \). This is well suited for dealing with sections of \( L_p \), since by a classical result of P. Lévy ([26]), \( L \in SL_p^n \) for \( p \geq 1 \) iff there exists a non-negative Borel measure \( \mu_L \) on \( S^{n-1} \) such that:

\[ \| x \|_L^p = \int_{S^{n-1}} |\langle x, \theta \rangle|^p d\mu_L(\theta) , \]

for all \( x \in \mathbb{R}^n \). This characterization extends to any \( p > 0 \), and it will enable us to extend the bound on \( L_K \) to the case \( K \in SL_p^n \) for all \( p > 0 \). As we shall see, for a general convex body \( K \), it is not the volume-ratio between \( L \in SL_p^n \) containing \( K \) and \( K \) which matters, but rather some other natural parameter. Moreover, our new technique will enable us to pass to the dual, and recover Junge’s bound on the isotropic constant of quotients of \( L_q \). In Section 5, we continue to apply our technique to bound the isotropic constant of convex bodies contained in \( k \)-Busemann-Petty bodies.
Theorem 4.1. Let $K$ be a centrally symmetric convex body in isotropic position, and let $D$ be a Euclidean ball normalized so that $\text{Vol}(D) = \text{Vol}(K)$. Then for any $p > 0$ and any $L \in SL^n_p$:

$$C_1/\sqrt{p_0} \leq L_K / \left( \frac{\tilde{V}_{-p}(L, K)}{V_{-p}(L, D)} \right)^{1/p} \leq C_2 \sqrt{p_0},$$

where $p_0 = \max(1, \min(p, n))$.

Remark 4.1. By taking the limit in (4.1) as $p \to 0^+$, we may define $SL^n_0$ to be the class of $n$-dimensional star-bodies $L$ for which:

$$\|x\|_L = \exp \left( \int_{S^{n-1}} \log \|x, \theta\| d\mu_L(\theta) + C \right),$$

for some Borel probability measure $\mu_L$ and constant $C$ and all $x \in \mathbb{R}^n$. In that case, Theorem 4.1 holds true for $p = 0$ as well (by passing to the limit), if we replace the expressions of the form $\tilde{V}_{-p}(L_1, L_2)^{1/p}$ appearing in (4.2), by the limit as $p \to 0^+$ assuming $\text{Vol}(L_2) = 1$, namely $\exp \left( \frac{1}{n} \int_{S^{n-1}} \log(\rho_{L_2}(x)/\rho_{L_1}(x)) \rho_{L_2}(x) dx \right)$.

Proof of Theorem 4.1. Let $\mu_L$ denote the Borel measure on $S^{n-1}$ from (4.1) corresponding to $L$. Then for any star-body $G$:

$$\tilde{V}_{-p}(L, G) = \frac{1}{n} \int_{S^{n-1}} \|x\|^p_G \|x\|_G^{-(n+p)} dx$$

$$= \frac{1}{n} \int_{S^{n-1}} \int_{S^{n-1}} |\langle x, \theta \rangle|^p d\mu_L(\theta) \|x\|_G^{-(n+p)} dx$$

$$= \frac{1}{n} \int_{S^{n-1}} d\mu_L(\theta) \int_{S^{n-1}} |\langle x, \theta \rangle|^p \|x\|_G^{-(n+p)} dx$$

$$= \frac{n + p}{n} \int_{S^{n-1}} d\mu_L(\theta) \int_G |\langle x, \theta \rangle|^p dx \quad (4.3)$$

Let us evaluate the expression $\int_G |\langle x, \theta \rangle|^p dx$. If $G$ is of volume 1 and $p \geq 1$, then by Jensen’s inequality:

$$\int_G |\langle x, \theta \rangle|^p dx \leq \left( \int_G |\langle x, \theta \rangle|^p dx \right)^{1/p} \quad \forall p \geq 1. \quad (4.4)$$

If $G$ is in addition convex, then by a well known consequence of a lemma by C. Borell ([?]), it follows that the linear functional $\langle \cdot, \theta \rangle$ has a $\psi_1$-type behaviour on $G$, and therefore:

$$\left( \int_G |\langle x, \theta \rangle|^p dx \right)^{1/p} \leq C_p \int_G |\langle x, \theta \rangle| dx \quad \forall p \geq 1 \quad (4.5)$$

If in addition $p \geq n$, it is well known that (e.g. [37, Lemma 4.1]):

$$\left( \int_G |\langle x, \theta \rangle|^p dx \right)^{1/p} \simeq \|\theta\|^p_G \quad \forall p \geq n. \quad (4.6)$$
Finally, if \( G \) is convex, of volume 1 and \( 0 < p < 1 \), then it follows from the estimates in Corollary 2.5 and 2.7 in [34] that:

\[
\left( \int_G |\langle x, \theta \rangle|^p \, dx \right)^{1/p} \simeq \int_G |\langle x, \theta \rangle| \, dx \quad \forall p \in (0, 1).
\]

The expression in (4.2) is invariant under simultaneous homothety of \( K \) and \( D \), so we may assume that \( \text{Vol} (K) = \text{Vol} (D) = 1 \). Since \( K \) is in isotropic position, we have \( \int_K \langle x, \theta \rangle^2 \, dx = L_K^2 \) for all \( \theta \in S^{n-1} \), and by (4.4) - (4.7) it follows that for all \( \theta \in S^{n-1} \):

\[
A \leq \left( \int_K |\langle x, \theta \rangle|^p \, dx \right)^{1/p} / L_K \leq B p_0 \quad \forall p > 0.
\]

It remains to notice that for a Euclidean ball \( D \) of volume 1, a straightforward computation (in the case \( 1 \leq p \leq n \)) together with (4.6) and (4.7), gives that for all \( \theta \in S^{n-1} \):

\[
\left( \int_D |\langle x, \theta \rangle|^p \, dx \right)^{1/p} \simeq \sqrt{p_0} \quad \forall p > 0.
\]

By (4.3), we have:

\[
\left( \frac{\tilde{V}_p(L, K)}{\tilde{V}_p(L, D)} \right)^{1/p} = \left( \frac{\int_{S^{n-1}} d\mu_L(\theta) \int_K |\langle x, \theta \rangle|^p \, dx}{\int_{S^{n-1}} d\mu_L(\theta) \int_D |\langle x, \theta \rangle|^p \, dx} \right)^{1/p}.
\]

Since \( \mu_L \geq 0 \), using (4.8) and (4.9), we get the required (4.2):

\[
\frac{1}{C_2 \sqrt{p_0}} \leq \left( \frac{\tilde{V}_p(L, K)}{\tilde{V}_p(L, D)} \right)^{1/p} / L_K \leq \frac{\sqrt{p_0}}{C_1}.
\]

\[
\square
\]

\textbf{Remark 4.2.} Notice that for \( 0 \leq p < 1 \), the unit-ball of a subspace of \( L_p \) is no longer necessarily a convex body. We will see more examples where \( L \) is a non-convex star-body later on. In fact, using the results in [20] of Guédon, it is possible to extend Theorem 4.1 to \( p > -1 \), but then the constants \( C_1 \) and \( C_2 \) will depend on \( p \). We do not proceed in this direction, because we are able to show in Section 5 that Theorem 4.1 is also valid for \( p = -1 \) (then \( SL^n_p \) is replaced by the class of intersection-bodies), and we are able to generalize this to \( k \)-Busemann-Petty bodies.

We can now extend Corollary 3.6 to the following more general result.

\textbf{Theorem 4.2.} Let \( K \) be a centrally symmetric convex body in isotropic position with \( \text{Vol} (K) = \text{Vol} (D_n) \). Then:

\[
L_K \leq C \inf \left\{ \frac{\sqrt{p_0}}{M_p(L)} \left| K \subseteq L, L \in SL^n_p, p \geq 0 \right. \right\},
\]

where \( p_0 = \max(1, \min(p, n)) \).
Proof. If \( K \subseteq L \), then obviously \( \tilde{V}_{-p}(L, K) \leq \tilde{V}_{-p}(K, K) = \text{Vol}(K) \). Applying Theorem 4.1 with \( \text{Vol}(D) = \text{Vol}(K) = \text{Vol}(D_n) \), (4.2) implies:

(4.10) \[
L_K \leq C_2 \sqrt{\rho_0} \left( \frac{\text{Vol}(D_n)}{\frac{1}{n} \int_{S^{n-1}} \rho_L(x)^{-p} dx} \right)^{1/p} = C_2 \sqrt{\rho_0} \frac{M_p(L)}{\text{Vol}(L)}.
\]

Using Jensen’s inequality (1.3) and homogeneity, we immediately have the following corollary, which unifies the bounds on \( L_K \) for \( SL_p^n \) of Ball (the case \( 1 \leq p \leq 2 \)) and Junge (the case \( p \geq 2 \)), and extends their results to \( p \geq 0 \):

**Corollary 4.3.** For any centrally symmetric convex body \( K \):

\[
L_K \leq C \inf \left\{ \sqrt{\rho_0} \left( \frac{\text{Vol}(L)}{\text{Vol}(K)} \right)^{1/n} \left| K \subseteq L \right., \ L \in SL_p^n, \ p \geq 0 \right\},
\]

where \( \rho_0 = \max(1, \min(p, n)) \).

**Remark 4.3.** Notice that the proof of Theorem 4.1 does not use the assumption that the body \( D \) is a Euclidean ball: the only property used is the one in (4.9). In fact, for the right-hand inequality in (4.2), \( D \) may be chosen as any \( \psi_2 \)-body in isotropic position. Recall that \( D \) is called a \( \psi_2 \)-body (with constant \( A > 1 \)), if for all \( p \geq 1 \):

\[
\left( \int_D |\langle x, \theta \rangle|^p dx \right)^{1/p} \leq A \sqrt[p]{p} \left( \int_D |\langle x, \theta \rangle|^2 dx \right)^{1/2} \forall \theta \in S^{n-1}.
\]

Bourgain has shown in [8] that if \( D \) is a \( \psi_2 \)-body then \( L_D \leq CA \log A \). Therefore if \( D \) is a \( \psi_2 \)-body of volume 1 in isotropic position, (4.9) may be replaced by:

\[
\left( \int_D |\langle x, \theta \rangle|^p dx \right)^{1/p} \leq A^2 \log A \sqrt[p]{p} \forall \theta \in S^{n-1}, \forall p \geq 1.
\]

(4.10) then reads (when \( \text{Vol}(K) = \text{Vol}(D) = \text{Vol}(D_n) \)):

\[
L_K \leq C(A) \sqrt{\rho_0} \left( \frac{\text{Vol}(D_n)}{\text{Vol}(L, D)} \right)^{1/p} = C(A) \sqrt{\rho_0} \left( \int_{S^{n-1}} \|x\|^p \rho_D(x)^{n+p} d\sigma(x) \right)^{-1/p}.
\]

By Bourgain’s result, if all linear functionals are \( \Psi_2 \), then \( L_K \) is bounded. Ironically, it follows from the proof of Theorem 4.1 that if all linear functionals have “bad” \( \psi_2 \) behaviour, e.g.

\[
\left( \int_K |\langle x, \theta \rangle|^q dx \right)^{1/q} \geq C \sqrt{q} \int_K |\langle x, \theta \rangle| dx \forall \theta \in S^{n-1},
\]

for a certain \( q \geq 1 \), then the bound on \( L_K \) improves \( (L_K \leq C \left( \frac{\text{Vol}(L)}{\text{Vol}(K)} \right)^{1/n} \) for all \( L \in SL_q^n \) containing \( K \), in the example above). Perhaps this may be used to our advantage?

We now turn to reproduce Junge’s bound on \( L_K \) for quotients of \( L_q \). As mentioned in the Introduction, for \( 1 < q \leq 2 \), Junge’s result is more general than ours and applies to all *subspaces* of quotients of \( L_q \). Nevertheless, our proof provides a (formally) stronger bound, applies to the entire range \( 1 < q \leq \infty \), and retains the problem’s Geometric nature, avoiding
unnecessary tools from Operator Theory. In addition, although derived independently, our proof is very similar to Bourgain’s proof that $L_K \leq Cn^{1/4}\log(1+n)$, and the latter may be thought of as an extremal case of our proof, where our argument breaks down.

**Theorem 4.4.** Let $K$ be a centrally symmetric convex body in isotropic position with $\text{Vol}(K) = \text{Vol}(D_n)$. Then:

$$L_K \leq C \inf \left\{ \sqrt[p_0]{M_p^*(T(L))} \left| \frac{K \subset L, L \in QL^n_q, T \in SL(n)}{1 \leq q \leq \infty, 1/p + 1/q = 1} \right. \right\},$$

where $p_0 = \min(p, n)$ and $p = q^*$ is the conjugate exponent to $q$.

We postpone the proof of Theorem 4.4 for later. In order to see why this Theorem implies Junge’s bound for quotients of $L_q$, we will need the following lemma:

**Lemma 4.5.** Let $K$ be a convex body with $\text{Vol}(K) = \text{Vol}(D_n)$.

1. If $K \in SL^n_p$ for $1 \leq p \leq \infty$, then there exists a position of $K$ for which $M^*_p(K) \leq C \sqrt[p_0]{p_0}$, where $p_0 = \min(p, n)$.
2. If $K \in QL^n_q$ for $1 \leq q \leq \infty$, then there exists a position of $K$ for which $M^*_p(K) \leq C \sqrt[p_0]{p_0}$, for $p = q^* = q/(q-1)$ and $p_0$ as above.

Applying the second part of the lemma to the body $L$ from Theorem 4.4 and using homogeneity, we immediately have:

**Corollary 4.6.** For any centrally symmetric convex body $K$:

$$L_K \leq C \inf \left\{ p_0 \left( \frac{\text{Vol}(L)}{\text{Vol}(K)} \right)^{1/n} \left| \frac{K \subset L, L \in QL^n_q}{1 \leq q \leq \infty, 1/p + 1/q = 1} \right. \right\},$$

where $p_0 = \min(p, n)$.

**Proof of Lemma 4.5.** We will prove part 1. Part 2 then follows easily by duality, using the reverse Blaschke-Santalo inequality ([11]):

$$\left( \frac{\text{Vol}(K)}{\text{Vol}(D_n)} \right)^{1/n} \left( \frac{\text{Vol}(K^o)}{\text{Vol}(D_n)} \right)^{1/n} \geq c,$$

(4.11)

to ensure that the volume of $K^o$ is not too small.

The case $1 \leq p \leq 2$ is straightforward, since for this range it is well known that sections of $L_p$ have finite volume-ratio (for instance, because they have cotype-2 and using [11], or by [5]). Therefore, in John’s maximal volume ellipsoid position, $M_p(K) \leq b(K) \leq c$. We remark that it remains to prove the lemma for $2 \leq p \leq n$, since it is known that $M_p(K) \simeq M_n(K) \simeq b(K)$ for $p > n$ (e.g. [28]).

We will present three different proofs for the case $2 \leq p \leq n$, placing the body $K$ in three different positions. We note that the first two proofs actually prove a stronger statement: for any $K \in SL^n_p$ there exists a position in which $M_p(K) \leq C \sqrt[p_0]{p_0}/a(K)$. Since this formulation is volume free, we do not really need the reverse Blaschke-Santalo inequality to prove the dual second part of the lemma (for the range $1 \leq q \leq 2$). The third proof is an elementary consequence of Theorem 4.2, and appears also in Corollary 6.3.
(1) If \(2 \leq p \leq n\), then \(T_2(X_K) \leq C\sqrt{p}\) (by Kahane’s inequality), so by Corollary 3.4, if \(K\) is in Lowner’s minimal volume outer ellipsoid position, then \(M_2(K) a(K) \leq C\sqrt{p}\). Notice that in Lowner’s position, \(b(K) \leq \sqrt{n}/a(K)\). Since \(\text{Vol}(K) = \text{Vol}(D_n)\), we obviously have \(a(K) \geq 1\), implying that \(M_2(K) \leq C\sqrt{p}\) and \(b(K) \leq \sqrt{n}\). We now use a known result from [28], stating that \(M_p(K) \simeq \max(M_2(K), b(K)\sqrt{p}/\sqrt{n})\), which under our conditions implies \(M_p(K) \leq C\sqrt{p}\).

(2) By approximation, we may assume that \(K\) is a section of \(L_m^p\), for some large enough \(m\). We will put \(K\) in the Lewis position ([27]), as used in [5]. In this position, there exists a sequence of \(m\) unit vectors \(\{u_i\}\) and positive scalars \(\{c_i\}\), such that \(\|x\|^p_K = \sum_{i=1}^m c_i |\langle x, u_i \rangle|^p\) and such that \(\mu = \sum_{i=1}^m c_i \delta_{u_i}\) is an isotropic measure (see Section 3). In particular, \(\sum_{i=1}^m c_i = n\). An elementary computation shows that for \(2 \leq p \leq n\):

\[
M_p(K) = \left(\sum_{i=1}^m c_i \int_{S^{n-1}} |\langle \theta, u_i \rangle|^p d\sigma(\theta)\right)^{1/p} \simeq \left(\sum_{i=1}^m c_i\right)^{1/p} \frac{\sqrt{p}}{\sqrt{n}} = \frac{\sqrt{p}}{n^{1/2-1/p}}.
\]

But in this position, Hölder’s inequality shows that:

\[
|x|^2 = \sum_{i=1}^m c_i |\langle x, u_i \rangle|^2 \leq \left(\sum_{i=1}^m c_i\right)^{1-2/p} \left(\sum_{i=1}^m c_i |\langle x, u_i \rangle|^p\right)^{2/p} = n^{1-2/p} \|x\|_K^2,
\]

and therefore \(a(K) \leq n^{1/2-1/p}\). It follows that \(M_p(K) \leq C\sqrt{p}/a(K)\), as required.

(3) Put the body \(K\) in isotropic position, and apply Theorem 4.2 with \(L = K\). Using the well known fact that \(L_K\) is always bounded from below by a universal constant (e.g. [34]), we immediately have \(M_p(K) \leq C\sqrt{p_0}(\text{Vol}(K)/\text{Vol}(D_n))^{1/n}\), and this is valid for all \(p \geq 0\), with \(p_0 = \max(1, \min(p, n))\).

\[\square\]

Proof of Theorem 4.4. Let \(K\) be in isotropic position and assume \(\text{Vol}(K) = 1\). Fix \(q > 1\) and let \(L \in QL^n_\infty\) contain \(K\). By duality, \(L^0\), the polar body to \(L\), is a section of \(L_p\), and so is \(T(L^0)\) for any \(T \in SL(n)\). Applying Theorem 4.1, the left (!) hand side of (4.2) gives:

\[
(4.12) \quad L_K \sqrt{p_0}/C_1 \geq \left(\frac{\bar{V}_{-p}(T(L^0), K)}{\bar{V}_{-p}(T(L^0), D)}\right)^{1/p} \geq \left(\frac{\bar{V}_{-p}(T(K^0), K)}{\bar{V}_{-p}(T(K^0), D)}\right)^{1/p},
\]

for \(D\) the Euclidean ball of volume 1. Evaluating the numerator on the right using the trivial \(\|x\|_{T(K^0)} \|x\|_K \geq |\langle T^{-1}(x), x \rangle|\), we have that for any positive-definite \(T \in SL(n)\):

\[
\left(\frac{\bar{V}_{-p}(T(K^0), K)}{\bar{V}_{-p}(T(K^0), D)}\right)^{1/p} \geq \left(\frac{1}{n} \int_{S^{n-1}} \|x\|^p_{T(K^0)} \|x\|^{-n+p}_K dx\right)^{1/p} \geq \left(\frac{n + 2p}{n} \int_K |\langle T^{-1}(x), x \rangle|^p dx\right)^{1/p} \geq \int_K |\langle T^{-1}(x), x \rangle| dx \geq \text{tr}(T^{-1}) L_K^2 \geq \det(T^{-1})^{1/n} L_K^2 = n L_K^2,
\]

where \(L_K^2 = \text{vol}(K)^2\) is the volume of \(K\).
where we have used Jensen’s inequality, the fact that \( \int_K x_i x_j dx = L_K^2 \delta_{i,j} \), and the Arithmetic-Geometric means inequality (since \( T \) is positive-definite). Together with (4.12), and cancelling out one \( L_K \) term, this gives:

\[
L_K \leq \frac{\sqrt{p_0}}{C_{1,n}} \left( \tilde{V}_{-p}(T(L^\circ), D) \right)^{1/p} \\
= \frac{\sqrt{p_0}}{C_{1,n}} \text{Vol}(D_n)^{-1/n} M_p(T(\mathbb{L})) \approx \frac{\sqrt{p_0}}{\sqrt{n}} M_p^*(\mathbb{L}),
\]

for any \( T \in SL(n) \) (since it can be factorized into a composition of a rotation and a positive-definite transformation, and \( M_p \) is invariant to rotations). Changing normalization from \( \text{Vol}(K) = 1 \) to \( \text{Vol}(K) = \text{Vol}(D_n) \), we have the desired:

\[
L_K \leq C \sqrt{p_0} M_p^*(T(L)).
\]

\( \square \)

**Remark 4.4.** As already mentioned, the proof of Theorem 4.4 clearly resembles Bourgain’s proof that \( L_K \leq C n^{1/4} \log(1 + n) \). In this respect, we mention that instead of using \( \sqrt{p_0} \) on the left hand side of (4.2) or \( \sqrt{p} \) on the left hand side of (4.12), it is easy to check that one may use \( A \), if \( K \) is a \( \Psi_2 \) body with constant \( A \) (as defined in Remark 4.3). This implies that whenever \( A < \sqrt{p} \), we get a better bound on \( L_K \). Bourgain has shown that in the general case, one may always assume that \( A \leq n^{1/4} \), but this does not seem to help us in our context.

To conclude this section, we mention that for a general convex body \( K \) (not necessarily a section of \( L_p \)), representations other than (4.1) of \( \| \cdot \|_K \) as a spherical convolution of a kernel with a non-negative Borel measure on \( S^{n-1} \) are known. Repeating the relevant parts of the proof of Theorem 4.1 with \( L = K \), it may be possible to bound some natural parameter of the body \( K \) other than \( L_K \).

### 5. \( k \)-Busemann-Petty bodies

An analogous result to Theorem 4.1 for \( k \)-Busemann-Petty bodies is the following:

**Theorem 5.1.** Let \( K \) be a centrally symmetric convex body in isotropic position, and let \( D \) be a Euclidean ball normalized so that \( \text{Vol}(D) = \text{Vol}(K) \). Then for any integer \( k = 1, \ldots, n - 1 \) and any \( L \in \mathcal{B}^n_k \):

\[
C_1 \leq L_K \left( \frac{\tilde{V}_k(L, D)}{\tilde{V}_k(L, K)} \right)^{1/k} \leq C_2 L_k.
\]

**Proof.** By definition, if \( L \in \mathcal{B}^n_k \) there exists a Borel measure \( \mu_L \) on \( G(n, n - k) \) such that:

\[
\rho_k^L = R_{n-k}^*(d\mu_L).
\]
Therefore, for any star-body $G$:

$$V_k(L, G) = \frac{1}{n} \int_{S^{n-1}} \rho_L(x)^k \rho_G(x)^{n-k} dx$$

$$= \text{Vol} (D_n) \int_{S^{n-1}} R_{n-k}^*(d\mu_L)(x) \rho_G(x)^{n-k} d\sigma(x)$$

$$= \text{Vol} (D_n) \int_{G(n,n-k)} R_{n-k}(\rho_G^{n-k})(E) d\mu_L(E)$$

$$= \frac{\text{Vol} (D_n)}{\text{Vol} (D_{n-k})} \int_{G(n,n-k)} \text{Vol} (G \cap E) d\mu_L(E).$$

The expression in (5.1) is invariant under simultaneous homothety of $K$ and $D$, so we may assume that $\text{Vol} (K) = \text{Vol} (D) = 1$. It is known ([1],[34],[4]) that for a convex $K$ in isotropic position and volume 1:

$$A \leq \text{Vol} (K \cap E)^{1/k} L_K \leq B \text{L}_k \quad \forall E \in G(n,n-k).$$

The proof of (5.4) is based on the fact that the function $f(x) = \text{Vol} (K \cap \{E + x\})$ on $E^+$ is log-concave and isotropic, and its isotropic constant is $L_f = f(0)^{1/k} L_K$. It was shown in ([1]) that an isotropic log-concave function $f$ on $\mathbb{R}^k$ satisfies $A \leq L_f \leq B \text{L}_k$, implying (5.4).

It remains to notice that for a Euclidean ball $D$ of volume 1, a straightforward computation shows that for any $k = 1, \ldots, n - 1$:

$$\text{Vol} (D \cap E)^{1/k} \simeq 1 \quad \forall E \in G(n,n-k).$$

By (5.3), we have:

$$\left( \frac{\tilde{V}_k(L, K)}{\tilde{V}_k(L, D)} \right)^{1/k} = \left( \frac{\int_{G(n,n-k)} \text{Vol} (K \cap E) d\mu_L(E)}{\int_{G(n,n-k)} \text{Vol} (D \cap E) d\mu_L(E)} \right)^{1/k}.$$

Since $\mu_L \geq 0$, using (5.4) and (5.5), we get the required (5.1):

$$C_1 \leq \left( \frac{\tilde{V}_k(L, K)}{\tilde{V}_k(L, D)} \right)^{1/k} L_K \leq C_2 \text{L}_k.$$

Remark 5.1. It is known ([25]) that the representation (5.2) exists for any star-body $L$ whose radial function $\rho_L$ is infinitely times differentiable on $S^{n-1}$, if we allow $\mu_L = \mu_{L,k}$ to be a signed measure on $G(n,n-k)$. Using $L = K$ for example, and repeating the argument in the proof of Theorem 5.1, we get that:

$$L_K \leq C \left( \frac{\int_{G(n,n-k)} |d\mu_{K,k}|(E)}{\int_{G(n,n-k)} d\mu_{K,k}(E)} \right)^{1/k} \text{L}_k,$$

so it remains to evaluate the above ratio. Unfortunately, this approach does not seem promising, since for a general smooth function $f$ on $S^{n-1}$, for which the representation $f = R_{n-k}^*(d\mu)$ is known to exist, it is easy to show that this ratio may be arbitrarily large for $k = 1$ and a fixed value of $n$.

We can now prove analogous results to Theorem 4.2 and Corollary 4.3.
Theorem 5.2. Let $K$ be a centrally symmetric convex body in isotropic position with $\text{Vol}(K) = \text{Vol}(D_n)$. Then:

$$L_K \leq C \inf \left\{ \mathcal{L}_k \tilde{M}_k(L) \mid K \subset L, \ L \in \mathcal{B} \mathcal{P}_k^n, \ k = 1, \ldots, n - 1 \right\}.$$ 

Proof. If $K \subset L$, then obviously $\tilde{V}_k(L, K) \geq \tilde{V}_k(K, K) = \text{Vol}(K)$. Applying Theorem 5.1 with $\text{Vol}(D) = \text{Vol}(K) = \text{Vol}(D_n)$, (5.1) implies:

$$L_K \leq C_2 L_k \left( \frac{\frac{1}{n} \int_{S^{n-1}} \rho_L(x)^k dx}{\text{Vol}(D_n)} \right)^{1/k} = C_2 L_k \tilde{M}_k(L).$$

□

Using Jensen’s inequality (1.3) and homogeneity, we immediately have the following corollary, which generalizes Ball’s bound on $L_K$ for $\text{SL}_p^n$ with $1 \leq p \leq 2$, since in that range $\text{SL}_p^n \subset \mathcal{B} \mathcal{P}_k^n$ for $k = 1, \ldots, n - 1$ (as explained in the Introduction):

Corollary 5.3. For any centrally symmetric convex body $K$:

$$L_K \leq C \inf \left\{ \mathcal{L}_k \left( \frac{\text{Vol}(L)}{\text{Vol}(K)} \right)^{1/n} \mid K \subset L, \ L \in \mathcal{B} \mathcal{P}_k^n, \ k = 1, \ldots, n - 1 \right\}.$$ 

Remark 5.2. As before, the proof of Theorem 5.1 does not utilize the assumption that $D$ is a Euclidean ball. The only property of $D$ used is the one stated in (5.5). By a result of Junge ([22]), this is satisfied by any 1-unconditional convex body in isotropic position. (5.6) then reads (when $\text{Vol}(K) = \text{Vol}(D) = \text{Vol}(D_n)$):

$$L_K \leq C_2 L_k \left( \frac{\tilde{V}_k(L, D)}{\text{Vol}(D_n)} \right)^{1/k} = C_2 L_k \left( \int_{S^{n-1}} \rho_L(x)^k \rho_D(x)^{n-k} d\sigma(x) \right)^{1/k}.$$ 

As in the previous section, we may prove dual counterparts to Theorem 5.2 and Corollary 5.3. Before proceeding, we will need the following useful lemma:

Lemma 5.4. For any compact set $A \subset \mathbb{R}^n$ and $m = 1, \ldots, n$:

$$\int_{G(n,m)} \text{Vol}(A \cap E) d\nu(E) \leq \inf_{T \in \text{SL}(n)} \sup_{E \in G(n,m)} \text{Vol}(T(A) \cap E),$$

where $\nu$ is the Haar probability measure on $G(n,m)$.

Proof. Notice that for any compact set $A \subset \mathbb{R}^n$ and $T \in \text{SL}(n)$:

$$\text{Vol}(A \cap E) = D_T(E) \text{Vol}(T(A) \cap T(E)),$$

where the Jacobian $D_T(E)$ does not depend on $A$. Now let $D$ be the Euclidean ball of volume 1, fix $T \in \text{SL}(n)$, and denote $G = G(n,m)$ for short. Denote $M = \sup_{E \in G} \text{Vol}(T(A) \cap E)$. 

Then:
\[
\int_G \text{Vol} (A \cap E) \, d\nu(E) = \int_G \text{Vol} (T(A) \cap T(E)) \, D_T(E) \, d\nu(E)
\]
\[
\leq M \int_G D_T(E) \, d\nu(E) = M \frac{\text{Vol} (D_n)^{m/n}}{\text{Vol} (D_m)} \int_G \text{Vol} (D \cap T(E)) \, D_T(E) \, d\nu(E)
\]
\[
= M \frac{\text{Vol} (D_n)^{m/n}}{\text{Vol} (D_m)} \int_G \text{Vol} (T^{-1}(D) \cap E) \, d\nu(E).
\]

Now, using polar coordinates, double integration and Jensen’s inequality, we have:
\[
\int_G \text{Vol} (T^{-1}(D) \cap E) \, d\nu(E) = \text{Vol} (D_m) \int_G \int_{S^{n-1} \cap E} \|\theta\|^{-m/n}_{T^{-1}(D)} \, d\sigma(\theta) \, d\nu(E)
\]
\[
= \text{Vol} (D_m) \int_{S^{n-1}} \|\theta\|^{-m/n}_{T^{-1}(D)} \, d\sigma(\theta) \leq \text{Vol} (D_m) \left( \int_{S^{n-1}} \|\theta\|^{-n}_{T^{-1}(D)} \, d\sigma(\theta) \right)^{\frac{m}{n}}
\]
\[
= \text{Vol} (D_m) \left( \frac{\text{Vol} (T^{-1}(D))}{\text{Vol} (D_n)} \right)^{\frac{m}{n}} = \frac{\text{Vol} (D_m)}{\text{Vol} (D_n)^{\frac{m}{n}}}. 
\]

We therefore see that for any $T \in SL(n)$:
\[
\int_{G(n,m)} \text{Vol} (A \cap E) \, d\nu(E) \leq \sup_{E \in G} \text{Vol} (T(A) \cap E),
\]
which proves the assertion. \hfill \square

**Remark 5.3.** An alternative way to prove Lemma 5.4 was suggested to us by the referee, to whom we are grateful. It makes use of a very interesting result by Grinberg ([18]), which was unknown to this author. In hope of interesting the unfamiliar reader, we bring it here. The dual affine Quermassintegral of a compact set $A$, which was introduced by Lutwak in the 80’s (see also [30]), is defined (up to normalization) as:
\[
\Phi_{n-m}(A) = \left( \int_{G(n,m)} \text{Vol} (A \cap E)^n \, d\nu(E) \right)^{1/n}.
\]

It was shown in [18] that $\Phi_{n-m}$ is indeed invariant to volume preserving linear transformations: $\Phi_{n-m}(T(A)) = \Phi_{n-m}(A)$ for all $T \in SL(n)$. Using this, Lemma 5.4 is easily deduced from Jensen’s inequality, since for any $T \in SL(n)$:
\[
\int_{G(n,m)} \text{Vol} (A \cap E) \, d\nu(E) \leq \left( \int_{G(n,m)} \text{Vol} (A \cap E)^n \, d\nu(E) \right)^{1/n}
\]
\[
= \Phi_{n-m}(A) = \Phi_{n-m}(T(A)) \leq \sup_{E \in G(n,m)} \text{Vol} (T(A) \cap E).
\]

We mention another result from [18], stating that for a convex body $K$:
\[
\Phi_{n-m}(K) \leq C_{m,n} \text{Vol} (K)^{m/n},
\]
where $C_{m,n}$ is determined by choosing $K = D_n$, and with equality iff $K$ is a centrally symmetric ellipsoid. This may be used to give a universal bound for the expression appearing
in the next Lemma 5.5, but we will need an estimate depending on \( L_K \) for the proof of Theorem 5.6.

Applying Lemma 5.4 on a convex body \( K \) of volume 1, and using (5.4) when \( T(K) \) is in isotropic position, we immediately get the following lemma as a corollary:

**Lemma 5.5.** For any centrally symmetric convex body \( K \) with \( \text{Vol} \,(K) = 1 \):

\[
\left( \int_{G(n,n-k)} \text{Vol} \,(K \cap E) \,d\nu(E) \right)^{1/k} \leq C \mathcal{L}_k / L_K,
\]

where \( \nu \) is the Haar probability measure on \( G(n, n-k) \).

We can now formulate the dual counterpart to Theorem 5.2. Note that since \( (L^o)^o \neq L \) for a general \( k \)-Busemann-Petty body, our formulation is a little different than before.

**Theorem 5.6.** Let \( K \) be a centrally symmetric convex body in isotropic position with \( \text{Vol} \,(K) = \text{Vol} \,(D_n) \). Then:

\[
L_K \leq C \inf \left\{ \frac{\mathcal{L}_2^2}{M_k(T(L))} \left| \begin{array}{l}
L \subset K^o, \ L \in \mathcal{BP}_k, \\
T \in \text{SL}(n), \ k = 1, \ldots, \lfloor n/3 \rfloor
\end{array} \right. \right\}.
\]

**Proof.** First, let us assume \( \text{Vol} \,(K) = 1 \), and correct for this later. Fix \( k = 1, \ldots, \lfloor n/3 \rfloor \) and let \( L \in \mathcal{BP}_k \) be contained in \( K^o \). As in the proof of Theorem 4.4, we note that \( T(L) \in \mathcal{BP}_k \) for any \( T \in \text{SL}(n) \). Applying Theorem 5.1, the left hand side of (5.1) gives:

\[
(5.7) \quad L_K / C_1 \geq \left( \frac{\tilde{V}_k(T(L), D)}{\tilde{V}_k(T(L), K)} \right)^{1/k} \geq \left( \frac{\tilde{V}_k(T(L), D)}{\tilde{V}_k(T(K^o), K)} \right)^{1/k},
\]

for \( D \) the Euclidean ball of volume 1. Evaluating the denominator on the right using the trivial \( \|x\|_{T(K^o)} \|x\|_K \geq |\langle T^{-1}(x), x \rangle| = |T^{-1/2}(x)|^2 \) for any positive definite \( T \in \text{SL}(n) \), we have that:

\[
\left( \frac{\tilde{V}_k(T(K^o), K)}{\tilde{V}_k(T(L), K)} \right)^{1/k} = \left( \frac{1}{n} \int_{S^{n-1}} \|x\|_{T(K^o)}^{-k} \|x\|_K^{-(n-k)} \,dx \right)^{1/k} \leq \left( \frac{1}{n} \int_{S^{n-1}} \|x\|_{T^{1/2}(D_n)}^{-2k} \|x\|_K^{-(n-2k)} \,dx \right)^{1/k} = V_{2k}(T^{1/2}(D_n), K)^{1/k}.
\]

Using property (2.1) of dual mixed-volumes, the latter expression is equal to \( V_{2k}(D_n, T^{-1/2}(K))^1/k \). Denoting \( G = G(n, n-k) \), and using polar coordinates and double integration, we have:

\[
V_{2k}(D_n, T^{-1/2}(K))^{1/k} = \left( \text{Vol} \,(D_n) \int_G \int_{S^{n-1} \cap E} \|\theta\|_{T^{-1/2}(K)}^{-n-2k} \,d\sigma_E(\theta) \,d\nu(E) \right)^{1/k} \leq C \left( \frac{\mathcal{L}_2}{L_K} \right)^2,
\]

where we have used Lemma 5.5 in the last inequality and (2.2). Together with (5.7), cancelling out one \( L_K \) term, and using \( n - 2k \geq n/3 \), this gives:

\[
(5.8) \quad L_K \leq C' n^{-1} \frac{\mathcal{L}_2^2}{\tilde{V}_k(T(L), D)}^{1/k} \simeq n^{-1/2} \frac{\mathcal{L}_2^2}{M_k(T(L))},
\]
for any $T \in SL(n)$ (since it can be factorized into a composition of a rotation and a positive-definite transformation, and $M_k$ is invariant to rotations). Now correcting for our initial assumption on $\text{Vol}(K)$ and going back to $\text{Vol}(K) = \text{Vol}(D_n)$, we have the desired:

$$L_K \leq C \frac{\mathcal{L}_{2k}^2}{M_k(T(L))}.$$

□

As in the previous section, it would be nice to know that for $L \in BP_n^k$, there exists a position in which we can bound $M_k(T(L))$ from below by $(\text{Vol}(L)/\text{Vol}(D_n))^{1/n}$ times some function of $k$. Unfortunately, we cannot provide an analogue of Lemma 4.5 for general $k$-Busemann-Petty bodies, but for convex members we have the following lemma, which is stated again in Corollary 6.3:

**Lemma 5.7.** Let $K$ be an isotropic convex body with $\text{Vol}(K) = \text{Vol}(D_n)$, and assume that $K \in BP_n^k$ for some $k = 1, \ldots, n - 1$. Then:

$$\overline{M}_k(K) \geq C/\mathcal{L}_k.$$

**Proof.** This is a trivial consequence of Theorem 5.2 applied with $L = K$, and using the well known fact (e.g. [34]) that $L_K$ is always bounded from below by a universal constant. □

We will therefore require that the body $L$ from Theorem 5.6 be convex, and denote by $CBP_n^k$ the class of convex $k$-Busemann-Petty bodies in $\mathbb{R}^n$. Applying Lemma 5.7 to the body $L$, using the reverse Blaschke-Santalo inequality (4.11) and homogeneity, we immediately have:

**Corollary 5.8.** For any centrally symmetric convex body $K$:

$$L_K \leq C \inf \left\{ \mathcal{L}_{2k}^2 \mathcal{L}_k \left( \frac{\text{Vol}(L)}{\text{Vol}(K)} \right)^{1/n} \middle| K \subset L, L \in CBP_n^k, k = 1, \ldots, \lfloor n/3 \rfloor \right\}.$$

We will see some applications of Theorem 5.2 in the next section.

**6. Applications**

As applications, we state a couple of immediate consequences of Corollaries 4.3 and 4.6 about the isotropic constant of polytopes with few facets or vertices. Next, we give several corollaries of Theorem 5.2, and show how they may be used to bound the isotropic constant of new classes of bodies.

It is well known that any centrally symmetric polytope with $2m$ facets is a section of an $m$-dimensional cube, and by duality, any centrally symmetric polytope with $2m$ vertices is a projection of an $m$-dimensional unit ball of $l_1$. It is also well known that $l_\infty$ isomorphically embeds in $L_p$ for $p = \log(1 + m)$, and by duality, $l_1^m$ is isomorphic to a quotient of $L_q$, for $q = p^*$ the conjugate exponent to $p$. With the same notations, it follows that a polytope with $2m$ facets is isomorphic to a section of $L_p$ and that a polytope with $2m$ vertices is isomorphic to a quotient of $L_q$. The following is therefore an immediate consequence of Corollary 4.3 or Junge’s Theorem:
Corollary 6.1. Let $K$ be a convex centrally symmetric polytope with $2m$ facets. Then $L_K \leq C \sqrt{\log(1 + m)}$.

Since any convex body may be isomorphically approximated by a polytope with $C^n$ facets (or vertices), we retrieve the well known naive bound $L_K \leq C \sqrt{n}$. In this respect, the factor of $\sqrt{p}$ in Corollary 4.3 for sections of $L_p$ seems more natural than the factor of $p$ for quotients of $L_q$, appearing in Corollary 4.6 or Junge’s Theorem. Reproducing the above argument, an immediate consequence of Corollary 4.6 or Junge’s Theorem is:

Corollary 6.2. Let $K$ be a convex centrally symmetric polytope with $2m$ vertices. Then $L_K \leq C \log(1 + m)$.

As mentioned in the Introduction, Corollary 6.2 implies that Gluskin’s probabilistic construction in [16] of two convex bodies $K_1$ and $K_2$ with Banach-Mazur distance of order $n$, satisfies $L_{K_1}, L_{K_2} \leq C \log(1 + n)$. This is simply because the bodies $K_1$ and $K_2$ are constructed as random polytopes with (at most) $4n$ vertices.

Another easy corollary, which was already partially stated in Lemmas 4.5 and 5.7, may be deduced from Theorems 3.5, 4.2 and 5.2, if we use the well known fact that $L_k$ may now consider as random polytopes with (at most) $4n$ vertices.

Corollary 6.3. Let $K$ be a convex centrally symmetric isotropic body with $\text{Vol}(K) = \text{Vol}(D_n)$. Then:

1. $1 \leq M_2(K) \leq CT_2(X_K)$.
2. If $K \in SL_p^n$ ($p \geq 0$), then $1 \leq M_p(K) \leq C \sqrt{p}$, where $p_0 = \max(1, \min(p, n))$.
3. If $K \in \mathcal{BP}_k^n$ ($k = 1, \ldots, n - 1$), then $C/L_k \leq M_k(K) \leq 1$.

Next, we proceed to deduce several consequences of Theorem 5.2. It is known that $\mathcal{BP}_k^n$ does not contain all convex bodies for $k < n - 3$, and that $\mathcal{BP}_{n-1}^n$ already contains all star-bodies ([12],[25]). So definitely not all convex bodies are isometric to members of $\mathcal{BP}_k^n$ for $k < n - 3$. Nevertheless, the following assumption might be true:

Outer Volume Ratio Assumption for $\mathcal{BP}_k^n$. There exist two universal constants $C, \epsilon > 0$, such that for any $n$ and any convex body $K$ in $\mathbb{R}^n$ there exists a star-body $L \in \mathcal{BP}_k^n$ for $k = n^{1-\epsilon}$, such that $K \subseteq L$ and $(\text{Vol}(L)/\text{Vol}(K))^{1/n} \leq C$.

Under this assumption, Theorem 5.2 would immediately imply that $L_n \leq C L_{n-\epsilon}$. Denoting $\delta = -1/\log(1 - \epsilon)$, and iterating this inequality $\delta \log \log n$ times, we would have:

Corollary 6.4. Under the Outer Volume Ratio Assumption for $\mathcal{BP}_k^n$, we have $L_n \leq C_1(\log(1 + n))^{C_2 \delta}$ for $\delta > 0$ as above.

In addition, the advantage of working with $\mathcal{BP}_k^n$ when trying to find or build a body $L \in \mathcal{BP}_k^n$ containing $K$, is that we need not worry about the convexity of $L$ like in the case of $SL_p^n$. The convexity of $K$ has already been used in Theorem 5.1 (in (5.4)), so we may now consider $\rho_K$ as a function on $S^{n-1}$ which we want to tightly bound from above using functions $\rho_L$ from the given family $\mathcal{BP}_k^n$. This is an especially attractive approach, as $\mathcal{BP}_k^n$ has the following nice characterization, first proved by Goodey and Weil in [17] for intersection-bodies (the case $k = 1$), and extended to general $k$ by Grinberg and Zhang in [19]:
**Theorem (Grinberg and Zhang).** A star-body \( K \) is a \( k \)-Busemann-Petty body iff it is the limit of \( \{ K_i \} \) in the radial metric \( d_r \), where each \( K_i \) is a finite \( k \)-radial sums of ellipsoids \( \{ E^i_j \} \):

\[
\rho_{K_i}^k = \rho_{E^i_1}^k + \cdots + \rho_{E^i_{m_i}}^k,
\]

or equivalently, if there exists a Borel measure \( \mu \) on \( SL(n) \) such that:

\[
\rho_{K_i}^k = \int_{SL(n)} \rho_{T(D_n)}^k d\mu(T).
\]

In fact, even the "easiest" case \( k = 1 \) in Theorem 5.2 seems potentially useful, as we shall demonstrate below. Note that since the intersection-body \( L \) need not be convex (and therefore Corollary 6.3 does not apply to it), the mean-radius \( M(L) \) might be significantly smaller than the volume-radius \( (\text{Vol}(L)/\text{Vol}(D_n))^{1/n} \). As demonstrated by Theorem 5.2, a smart way to bound \( \rho_K \) from above by \( \rho_L \) which is the sum of radial functions of ellipsoids, such that we have control over \( L \)'s mean-radius, might provide a new bound on the isotropic constant. We give two examples of how such an approach might work. Unfortunately, we need to use some additional assumptions, which, although we believe to be true, we have not been able to prove. First, we need a new definition for a class of bodies.

**Definition.** Let \( K \) denote a star-body. We will work with the radial metric topology on the space of star-bodies. Introduce the closed set of volume preserving linear images of \( K \),

\[
B(K) = \{ T(K) \mid T \in SL(n) \}.
\]

The *Radial Sums of \( K \)*, denoted by \( RS(K) \), is the closure in the radial metric of the family of all star-bodies \( L \), such that there exists a non-negative Borel measure \( \mu \) on \( B(K) \), for which:

\[
\rho_L = \int_{B(K)} \rho_{K'} d\mu(K').
\]

Similarly, if \( P \) is a closed set of star-bodies, then the *Radial Sums of \( P \)*, denoted \( RS(P) \), is the closure in the radial metric of the family of all star-bodies \( L \), such that there exists a non-negative Borel measure \( \mu \) on \( B(P) = \bigcup_{K \in P} B(K) \), for which:

\[
\rho_L = \int_{B(P)} \rho_{K'}(\theta) d\mu(K').
\]

So for example \( RS(D_n) \) is exactly the class of intersection-bodies, since \( B(D_n) \) is the set of all ellipsoids of volume \( \text{Vol}(D_n) \), and by the aforementioned result of Goodey and Weil, the radial sums of this set are exactly the class of intersection-bodies. Another easy observation is that \( RS(P) \) is closed under full-rank linear transformations, since for any linear \( T \):

\[
\rho_K = \rho_{K_1} + \rho_{K_2} \Rightarrow \rho_{T(K)} = \rho_{T(K_1)} + \rho_{T(K_2)}.
\]

As a consequence, \( RS(D_n) \subset RS(K) \) for any star-body \( K \). To see this, first notice that \( D_n \in RS(K) \), by choosing the Borel measure \( \mu \) on \( B(K) \) to be:

\[
\mu(A) = \eta(\{ T \in O(n) \mid T(K) \in A \})
\]
for every Borel set \( A \subset B(K) \), where \( \eta \) is the appropriately normalized Haar measure on \( O(n) \), the group of orthogonal rotations in \( \mathbb{R}^n \). Since \( RS(K) \) is closed under \( SL(n) \), radial summation, and limit in the radial-metric, it follows that \( RS(D_n) \subset RS(K) \). Therefore, for any non intersection-body \( K \), \( RS(K) \) properly contains the class of intersection bodies.

There are many interesting questions that may be asked about Radial Sums of star-bodies, such as whether it is possible to characterize a minimal set \( P \) for every Borel set \( A \subset B(K) \). As mentioned before, the families of convex bodies in \( RS(Q_n) \) are all unconditional with respect to the same fixed Euclidean structure. We shall say that \( \rho_\ast \) is an ellipsoid if it is a linear-image of the unit ball of \( l_1^n \).

Corollary 6.5. Under the Outer Mean-Radius Assumption for the Cube \( Q_n \), for any convex body \( K \in RS(Q_n) \), we have \( L_K \leq C \log(1 + n) \).

Under the Outer Mean-Radius Assumption for \( UC(n) \), for any convex body \( K \in RS(UC(n)) \), we have \( L_K \leq C \log(1 + n) \).

As mentioned before, the families of convex bodies in \( RS(Q_n) \) and \( RS(UC(n)) \) are potentially new classes of convex bodies, which might contain a big piece of the convex bodies compactum. Therefore, this new approach to bounding the isotropic constant might be applicable for a large family of convex bodies.

Proof. Let \( K \) be an isotropic convex body of volume \( Vol(D_n) \) in \( RS(P) \), where \( P \) is either \( \{Q_n\} \) or \( UC(n) \). By approximation, we may assume that \( \rho_K = \sum_i \mu_i \rho_{K_i} \), where \( K_i \in B(P) \) and \( \mu_i \geq 0 \).

Notice that both the unit-ball of \( l_1^n \) and the Euclidean ball are intersection bodies, and this is preserved under volume preserving linear transformations. Therefore, by the Outer Mean-Radius Assumption for \( P \), there exist intersection-bodies \( L_i \) such that \( K_i \subset L_i \) and \( \tilde{M}(L_i)/\tilde{M}(K_i) \leq C \log(1 + n) \). Now define \( L \) to be the star-body for which \( \rho_L = \sum_i \mu_i \rho_{L_i} \).

It is obvious that \( L \) contains \( K \), and that \( L \) is an intersection-body (since these are closed under non-negative radial summation, as follows from their definition). In addition, since the mean-radius \( \tilde{M} \) is additive under radial summation, it is clear that \( \tilde{M}(L)/\tilde{M}(K) \leq \)}
Proposition 6.6.

1. Let $D$ be the circumscribing Euclidean ball of $Q_n$. Then:
   \[ \tilde{M}(D)/\tilde{M}(Q_n) \leq C \log(1 + n). \]

2. Let $K$ be an unconditional convex body in isotropic position, and let $L$ be its circumscribing unit ball of $l_1^n$. Then:
   \[ \tilde{M}(L)/\tilde{M}(K) \leq C \log(1 + n). \]

Proof.

1. This is a standard calculation relating to the concentration of the norm $\| \cdot \|_{Q_n}$ on the sphere, which may be done using the standard concentration techniques from [36]. We prefer to quote a general result by Klartag and Vershynin from [23, Proposition 1.2], which states that for any convex body $K$, if $0 < l < Ck(K)$, where $k(K) = n(M(K)/b(K))^2$, then $\tilde{M}_l(K) \simeq 1/M(K)$. Since for the volume 1 cube $Q_n$ it is well known (e.g. [36]) that $M(Q_n) \simeq \sqrt{n}/\sqrt{n}$, $b(Q_n) = 2$, and therefore $k(Q_n) \simeq \sqrt{\log(1 + n)}$, it follows that for $n$ large enough we may use the above result for $l = 1 < Ck(Q_n)$, to conclude that (for all $n$) $\tilde{M}(Q_n) \simeq \sqrt{n}/\sqrt{\log(1 + n)}$. Since $\tilde{M}(D) = \sqrt{n}/2$, the claim follows.

2. Let $P_n$ be the unit ball of $l_1^n$ of volume 1. It is well known (e.g. [6]) that there exist $C_1, C_2 > 0$, such that for any isotropic convex body $K$ of volume 1, which is unconditional with respect to the given Euclidean structure, the following inclusions hold:
   \[ C_1 Q_n \subset K \subset C_2 P_n. \]
   Therefore $\tilde{M}(L)/\tilde{M}(K) \leq \tilde{M}(C_2 P_n)/\tilde{M}(C_1 Q_n)$. We have already seen that $\tilde{M}(Q_n) \simeq \sqrt{n}/\sqrt{\log(1 + n)}$. We may estimate $\tilde{M}(P_n)$ in the same manner, or alternatively, use Corollary 6.3 to deduce that $\tilde{M}(P_n) \simeq \sqrt{n}$. Therefore $\tilde{M}(L)/\tilde{M}(K) \leq C \log(1 + n)$. 

Unfortunately, the techniques described above fail when used upon $T(K)$, where $K$ is in isotropic position but $T$ is an almost degenerate mapping. In particular, it is a bad idea to try to bound $T(K)$ using $T(L)$, where $L$ is the optimal bounding body for $K$. Indeed, let us try to evaluate $\tilde{M}(T(D))/\tilde{M}(T(Q_n))$, where as in Proposition 6.6, $D$ is the circumscribing Euclidean ball of $Q_n$. Using (2.1), we have:

\[
\frac{\tilde{M}(T(D))}{M(T(Q_n))} = \frac{\tilde{V}_1(T(D), D_n)}{\tilde{V}_1(T(Q_n), D_n)} = \frac{\tilde{V}_1(D, T^{-1}(D_n))}{\tilde{V}_1(Q_n, T^{-1}(D_n))}.
\]
Denoting $\mathcal{E} = T^{-1}(D_n)$, we see that:
\[
\frac{\tilde{M}(T(D))}{\tilde{M}(T(Q_n))} = \frac{\int_{S^{n-1}} \rho_D(\theta) \rho_{\mathcal{E}}(\theta)^{n-1} d\sigma(\theta)}{\int_{S^{n-1}} \rho_{Q_n}(\theta) \rho_{\mathcal{E}}(\theta)^{n-1} d\sigma(\theta)},
\]
and this is clearly invariant under homothety of $\mathcal{E}$. Now let us define $\mathcal{E}(\xi, a, b)$ for $\xi \in S^{n-1}$ and $a, b > 0$ as the ellipsoid whose corresponding norm is defined as:
\[
\|x\|_{\mathcal{E}(\xi, a, b)}^2 = \frac{\langle x, \xi \rangle^2}{a^2} + \frac{|x|^2 - \langle x, \xi \rangle^2}{b^2}.
\]
It was shown in [17] that by appropriately choosing $a = a(\epsilon)$ very large and $b = b(\epsilon)$ very small, and setting $\mathcal{E}(\xi, \epsilon) = \mathcal{E}(\xi, a(\epsilon), b(\epsilon))$, the family $\rho_{\mathcal{E}(\xi, \epsilon)}^{n-1}$ is an approximation of unity on $S^{n-1}$ at $\xi$ (as $\epsilon > 0$ tends to 0). This means that for every $f \in C(S^{n-1})$:
\[
\int_{S^{n-1}} f(\theta) \rho_{\mathcal{E}(\xi, \epsilon)}^{n-1}(\theta) d\sigma(\theta) \longrightarrow f(\xi) \text{ as } \epsilon \to 0.
\]
Hence, we see that by choosing $T = T(\xi)$ to be very degenerate, we may arbitrarily approximate:
\[
\frac{\tilde{M}(T(D))}{\tilde{M}(T(Q_n))} \approx \frac{\rho_D(\xi)}{\rho_{Q_n}(\xi)},
\]
and the latter ratio may be chosen to be any number between 1 and $\sqrt{n}$ by an appropriate choice of $\xi \in S^{n-1}$. This example demonstrates the difficulty in proving the Outer Mean-Radius Assumptions.

**References**


E-mail address: emanuel.milman@weizmann.ac.il

DEPARTMENT OF MATHEMATICS, THE WEIZMANN INSTITUTE OF SCIENCE, REHOVOT 76100, ISRAEL.
CHAPTER 5

ON VOLUME DISTRIBUTION IN 2-CONVEX BODIES

BO'AZ KLARTAG♯ AND EMANUEL MILMAN†

to appear in Israel Journal of Mathematics

ABSTRACT. We consider convex sets whose modulus of convexity is uniformly quadratic. First, we observe several interesting relations between different positions of such “2-convex” bodies; in particular, the isotropic position is a finite volume-ratio position for these bodies. Second, we prove that high dimensional 2-convex bodies possess one-dimensional marginals that are approximately Gaussian. Third, we improve the known bounds on the isotropic constant of quotients of subspaces of $L_p$ and $S_m^n$, the Schatten Class space, for $1 < p \leq 2$.

1. Introduction

The purpose of this note is to collect several interesting facts related to the distribution of volume in high dimensional 2-convex bodies. Suppose that $K \subset \mathbb{R}^n$ is a centrally-symmetric (i.e. $K = -K$) convex body (i.e. a convex, compact set with non-empty interior). Let $\|\cdot\|_K$ be the norm on $\mathbb{R}^n$ whose unit ball is $K$. The modulus of convexity of $K$ is the function:

\[
\delta_K(\varepsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{\|x\|_K, \|y\|_K \leq 1, \|x - y\|_K \geq \varepsilon} \right\},
\]

defined for $0 < \varepsilon \leq 2$. We say that $K$ is “2-convex with constant $\alpha$” (see, e.g. [24, Chapter 1.e]), if for all $0 < \varepsilon \leq 2$,

\[
\delta_K(\varepsilon) \geq \alpha \varepsilon^2.
\]

Note that this should not be confused with the notions of $p$-convexity or $q$-concavity (e.g. [24, Chapter 1.d]) defined for Banach lattices. Being 2-convex with constant $\alpha$ is a linearly invariant property. Furthermore, as is evident from the definitions, if $K$ is 2-convex with constant $\alpha$, so is $K \cap E$ for any subspace $E$. Thus sections of a convex body inherit the 2-convexity properties of the body. The same holds for projections (see, e.g. Lemma 3.4 below). A basic example of 2-convex bodies are unit balls of $L_p$ spaces for $1 < p \leq 2$, in which case $\alpha$ is of the order of $p - 1$ (e.g. [24, Chapter 1.e]). Consequently, also sections, projections, and sections of projections of $L_p$-balls are 2-convex bodies, with constants that depend solely on $p$.

It is well-known that the uniform measure on a 2-convex body is “well behaved”, in many senses (see, e.g. [13] [36] and [5]). Questions on distribution of mass in high-dimensional

♯ supported by the Clay Mathematics Institute and by NSF grant #DMS-0456590.
† supported in part by BSF and ISF.
convex sets regained some interest in the last few years, and partial progress was obtained. We approach the study of mass distribution in 2-convex sets, in view of these developments. Arguably, the most basic question regarding volume distribution in high-dimensional convex sets is the Slicing Problem, or Hyperplane Conjecture. This question asks whether for any convex body $K \subset \mathbb{R}^n$ of volume one, there exists a hyperplane $H \subset \mathbb{R}^n$ such that $\text{Vol}(K \cap H) > c$, for some universal constant $c > 0$. Here and henceforth, $\text{Vol}(A)$ or $|A|$ for short, denotes the volume of $A \subset \mathbb{R}^n$ in its affine hull. In the category of 2-convex bodies, a positive answer to this question was provided by Schmuckenschläger [36]. We provide a more direct approach to Schmuckenschläger’s result, that is based on an argument of [2].

**Proposition 1.1.** Let $K \subset \mathbb{R}^n$ be a centrally-symmetric convex body of volume one. Suppose $K$ is 2-convex with constant $\alpha$. Then there exists a hyperplane $H \subset \mathbb{R}^n$ such that:

$$\text{Vol}(K \cap H) \geq c\sqrt{\alpha},$$

where $c > 0$ is a universal constant.

A centrally-symmetric convex $K \subset \mathbb{R}^n$ of volume one is said to be isotropic or in isotropic position, if for any $\theta \in \mathbb{R}^n$:

$$\int_K \langle x, \theta \rangle^2 dx = L_K |\theta|^2,$$

where $L_K$ is some quantity, independent of $\theta$, and $| \cdot |$ is the Euclidean norm. In that case, the isotropic constant of $K$ is defined as $L_K$. It is well known (see, e.g. [29]) that for any centrally-symmetric convex $K \subset \mathbb{R}^n$, there exists a linear transformation such that $\tilde{K} = T(K)$ is isotropic. Moreover, this map $T$ is unique up to orthogonal transformations.

We therefore define the isotropic constant of an arbitrary centrally-symmetric convex body $K \subset \mathbb{R}^n$, to be $L_K = L_{\tilde{K}}$, where $\tilde{K}$ is an isotropic linear image of $K$. An observation that goes back to Hensley [16], is that when $K$ is isotropic, for any hyperplane $H$ through the origin:

$$c_1 \frac{L_K}{c_2} \leq \text{Vol}(K \cap H) \leq \frac{c_2}{L_K},$$

where $c_1, c_2 > 0$ are universal constants. Based on this, the Slicing Problem may be reformulated as follows (e.g. [29]): Is it true that for any dimension $n$ and any centrally-symmetric convex body $K \subset \mathbb{R}^n$, we have that $L_K \leq C$, where $C > 0$ is a universal constant?

As a by-product of our methods, we improve the known bounds for the isotropic constant of the unit balls of quotients of subspaces of $L_p$ for $1 < p \leq 2$, and establish the same bound for arbitrary quotients of subspaces of $l_p$-Schatten-Class spaces of $m$ by $m$ matrices, denoted $S^m_p$ (see Section 3 for definitions). For a Banach Space $X$, we denote by $SQ_n(X)$ the family of all centrally-symmetric convex bodies $K \subset \mathbb{R}^n$, such that $K$ is the unit ball of some subspace of a quotient of $X$.

**Proposition 1.2.** Let $1 < p \leq 2$, let $X = L_p$ or $X = S^m_p$, and suppose that $K \in SQ_n(X)$. Then,

$$L_K \leq C \sqrt{q}$$

where $q = p^* = p/(p - 1)$ and $C > 0$ is a universal constant.
Junge [18] has previously proven a version of (1.3) with $q$ in place of $\sqrt{q}$ for $X = L_p$. For $X = S_p^m$ and $1 \leq p \leq 2$, a universal bound on $L_K$ was established in [22] when $K$ is the unit ball of $X$, and in [14] when $K$ is the unit ball of certain specific subspaces of $X$.

In addition to the isotropic position, there are several other important Euclidean structures that are associated with a given convex body, such as John’s position, minimal mean-width position, $\ell$-position, (regular) $M$-position, etc. The relations between these various positions in general are not clear. See [6] for an equivalence of the hyperplane conjecture to a certain putative relation between the isotropic position and $M$-position. However, in the class of 2-convex bodies, the following holds:

**Proposition 1.3.** Let $K \subset \mathbb{R}^n$ be a 2-convex body with constant $\alpha$ and of volume 1. If $K$ is in isotropic position then:

$$c\sqrt{\alpha}\sqrt{n}D_n \subset K,$$

where $D_n$ is the unit Euclidean ball in $\mathbb{R}^n$ and $c > 0$ is a universal constant.

That is, the isotropic position of a 2-convex body is a finite volume-ratio position. The volume-ratio of a centrally-symmetric convex body $K \subset \mathbb{R}^n$ is defined as:

$$v.r.(K) = \min_{E \subset K} \left( \frac{|K|}{|E|} \right)^{\frac{1}{n}},$$

where the minimum is taken over all ellipsoids that are contained in $K$. If $v.r.(K) < C$, for some universal constant $C$, it is customary to say that $K$ is a finite volume-ratio body. When the minimum over all Euclidean balls is bounded by a universal constant, we will say that $K$ is in a finite volume-ratio position. Note that $c_1 < |\sqrt{n}D_n|^{1/n} < c_2$ for some universal constants $c_1, c_2 > 0$, so Proposition 1.3 implies that the isotropic position is a finite volume-ratio position.

This conclusion is clearly false for general convex bodies, even for convex bodies whose distance to the Euclidean ball is universally bounded (see the example after Lemma 2.3 below). In Section 4 we establish further rigid relations between various positions of 2-convex bodies, that cannot hold for arbitrary convex bodies. In particular, recall that $K$ is said to be in John’s maximal-volume ellipsoid position when the minimum in (1.4) is attained by a Euclidean ball. We will see the following:

**Proposition 1.4.** Let $K \subset \mathbb{R}^n$ be a 2-convex body with constant $\alpha$ and of volume 1. If $K$ is in John’s maximal-volume ellipsoid position, then:

$$\left( \int_K |x|^2 \, dx \right)^{\frac{1}{2}} \leq C \frac{\sqrt{n}}{\alpha},$$

where $C > 0$ is a universal constant.

The latter is in a sense a converse to Proposition 1.3, since (1.5) implies that $K$ is “essentially” isotropic. To see this, note (e.g. [29]) that the isotropic position minimizes the value of $\int_{T(K)} |x|^2 \, dx$, over all volume 1 affine images $T(K)$ of $K$, and in that case we have:

$$\inf \left( \int_{T(K)} |x|^2 \, dx \right)^{\frac{1}{2}} = \sqrt{n}L_K.$$
In addition to being an “essentially” isotropic position, we show in Section 4 that John’s position is in fact an “essentially” minimal mean-width position and a 2-regular M-position (see Section 4 for definitions). A complete list of other relations between the aforementioned various positions is given at the end of Section 4.

An additional interesting volumetric question, is the so-called “Central Limit Property of Convex Bodies”. Let $X$ denote a uniformly distributed vector inside a convex set $K \subset \mathbb{R}^n$ of volume one. In its weakest form, a conjecture of Antilla, Ball and Perissinaki [1] and Brehm and Voigt [8], states that for some non-zero vector $\theta \in \mathbb{R}^n$, the random variable $\langle X, \theta \rangle$ is very close to a Gaussian random variable. That is, the total variation distance between the random variable $\langle X, \theta \rangle$ and a corresponding Gaussian random variable, is smaller than $\varepsilon_n$, where $\varepsilon_n$ is a sequence tending to zero, that depends solely on $n$. In this note, we verify the following (see Theorem 5.4 for an exact formulation):

**Proposition 1.5.** The “Central Limit Property” holds true for arbitrary 2-convex bodies.

In [1], the existence of approximately Gaussian marginals of 2-convex bodies was proven only under a certain, rather weak, constraint on the diameter of $K$ in isotropic position. We show in Example 4.9 that there exist 2-convex bodies in $\mathbb{R}^n$ for which this constraint is violated. In fact, we show that there exist such bodies of volume 1 whose diameter in isotropic position is greater than $cn$ (where $c > 0$ is a universal constant). Our idea is to put $K$ in another position, namely L"owner’s minimal diameter position, in which we show in Proposition 4.10 that the diameter is not larger than $Cn^{1-\lambda}$, where $\lambda$ depends only on $\alpha$, the 2-convexity constant of $K$ and $C > 0$ is a universal constant. We conclude Proposition 1.5 by using Theorem 5.3 taken from [28], which generalizes a Theorem from [1] about the existence of Gaussian marginals, by removing the assumption that $K$ is in isotropic position. Further developments on the existence of Gaussian marginals of uniformly convex bodies are discussed in [28].

The rest of the paper is organized as follows. In Section 2 we discuss the basic volumetric properties of 2-convex bodies. In Section 3 we consider natural operations which preserve 2-convexity and its dual notion of 2-smoothness, and prove generalized versions of Proposition 1.2. Section 4 treats various positions of 2-convex bodies and their interrelations. Section 5 deals with Gaussian marginals. Throughout the text, we denote by $c, C, c'$ etc. some positive universal constants, whose value may change from line to line. We will write $A \simeq B$ to signify that $C_1 A \leq B \leq C_2 A$ with universal constants $C_1, C_2 > 0$. We denote by $D_n$ and $S^{n-1}$ the Euclidean unit ball and sphere in $\mathbb{R}^n$, respectively.

**Acknowledgments.** Emanuel Milman would like to sincerely thank his supervisor Prof. Gideon Schechtman for many informative discussions.

### 2. Volumetric properties

Let $K \subset \mathbb{R}^n$ be a centrally-symmetric convex body. Denote by $\|\cdot\|_K$ the norm whose unit ball is $K$. The dual norm to $\|\cdot\|_K$ is defined as $\|x\|_K^* = \sup_{y \in K} |\langle x, y \rangle|$, and its unit ball, referred to as the polar body to $K$, is denoted by $K^o$. 
An equivalent well-known characterization for $K$ to be 2-convex with constant $\alpha$ (e.g. [24, Lemma 1.e.10]) is that for all $x, y \in \mathbb{R}^n$:

$$
\|x\|^2_K + \|y\|^2_K - 2\left\|\frac{x + y}{2}\right\|^2_K \geq \frac{\alpha'}{2}\|x - y\|^2_K, 
$$

where the relation between $\alpha$ and $\alpha'$ is summarized in the following:

**Lemma 2.1.** If $K$ is 2-convex with constant $\alpha$ then (2.1) holds with $\alpha' = \alpha$. If (2.1) holds for all $x, y \in \mathbb{R}^n$, then $K$ is 2-convex with constant $\alpha = \alpha' / 8$.

It is also known ([31]) that the Euclidean ball has the best possible modulus of convexity, implying in particular that $\alpha \leq 1/8$.

A basic observation due to Gromov and Milman ([13], see also [2] for a simple proof) is that if $K$ is uniformly convex with modulus of convexity $\delta_K$, and $T \subset K$ with $|T| \geq \frac{1}{2}|K|$, then for any $\varepsilon > 0$:

$$
\left|\frac{(T + \varepsilon K) \cap K}{|K|}\right| \geq 1 - 2e^{-2n\delta_K(\varepsilon)}. 
$$

We will exploit (2.2) and obtain several interesting consequences regarding mass distribution in 2-convex sets. At the heart of our argument is the following lemma, which is a direct consequence of (2.2). We prefer to give a self-contained proof, as this is a good opportunity to recreate the elegant argument from [2]. This lemma was also proved in [36].

**Lemma 2.2.** Let $K \subset \mathbb{R}^n$ be a centrally-symmetric convex body. Assume that $K$ is 2-convex with constant $\alpha$, and that $|K| = 1$. Fix $\theta \in S^{n-1}$ and denote $w = \|\theta\|^*_K$. Then for any $t > 0$:

$$
\text{Vol}\{x \in K; \langle x, \theta \rangle > t\} \leq 2 \exp\left(-2\alpha n\left(\frac{t}{w}\right)^2\right).
$$

**Proof.** Let $A(t) = \{x \in K; \langle x, \theta \rangle > t\}$ and put $B = \{x \in K; \langle x, \theta \rangle < 0\}$. Note that if $x \in A(t), y \in B$ then $\|x - y\|_K \geq \frac{t}{w}$. According to the definition of 2-convexity,

$$
\frac{B + A(t)}{2} \subset \left(1 - \alpha \left(\frac{t}{w}\right)^2\right)K.
$$

By the Brunn-Minkowski inequality,

$$
\sqrt{|B| \cdot |A(t)|} \leq \left|\frac{B + A(t)}{2}\right| \leq \left(1 - \alpha (t/w)^2\right)^{n} \leq \exp\left(-\alpha n (t/w)^2\right).
$$

Since $|B| = 1/2$, we have:

$$
|A(t)| \leq 2 \exp\left(-2\alpha n (t/w)^2\right).
$$

□

Next, we present several consequences of Lemma 2.2. The first one is the following observation.

**Lemma 2.3.** Let $K \subset \mathbb{R}^n$ be a centrally-symmetric convex body. Assume that $K$ is 2-convex with constant $\alpha$ and volume 1, and that $K$ is isotropic. Then:

$$
c\sqrt{\alpha} \sqrt{n}LD_n \subset K,
$$

where $c > 0$ is a universal constant.
Proof. Let $\theta \in S^{n-1}$ be arbitrary. For $t \in \mathbb{R}$ set
$$A(t) = K \cap \{ x \in \mathbb{R}^n; \langle x, \theta \rangle < t \},$$
and denote $f(t) = |A(t)|$. As before, we use $w = \|\theta\|_K^*$ to denote the width of $K$ in direction $\theta$. By Lemma 2.2, we see that for $t > 0$:

$$f(t) \geq 1 - 2 \exp\left(-2\alpha n(t/w)^2\right).$$

(2.3)

On the other hand, $f'(t) = |K \cap \{ \langle x, \theta \rangle = t \}|$ is a log-concave function by Brunn-Minkowski which is even, and therefore attains its maximum at 0. Since $f'(0) \simeq 1/L_K$ (e.g. [29]), we see that:

$$f(t) \leq f(0) + tf'(0) \leq \frac{1}{2} + \frac{t}{L_K}.$$

Choosing $t = L_K/4c$ and combining (2.3) and (2.4), we see that $w \geq c\sqrt{\alpha} \sqrt{n}L_K$. Since the direction $\theta \in S^{n-1}$ was arbitrary, the lemma follows. 

Lemma 2.3 entails Proposition 1.1 and Proposition 1.3 at once. Indeed, since $|\sqrt{n}D_n|^{1/n} \simeq 1$ and $|K| = 1$, Lemma 2.3 implies that $L_K \leq c/\sqrt{\alpha}$. Proposition 1.1 immediately follows (see, e.g. [29]). Since also $c < L_K$ (e.g. [29]), then Lemma 2.3 implies that:

$$c\sqrt{\alpha} \sqrt{n}D \subset K,$$

and Proposition 1.3 is established. Note that it is quite unusual for a convex body to contain a large Euclidean ball in isotropic position, even when the body has a bounded volume-ratio. For instance, consider the convex body $K = \{ x \in \mathbb{R}^n; |x| \leq \sqrt{n}, |x_1| \leq 1 \}$, and let $\tilde{K}$ be an isotropic linear image of $K$. It is easily seen that $\tilde{K}$ does not contain a ball of radius larger than $c$, although $K$ is isomorphic to a Euclidean ball, and clearly has a finite volume-ratio.

Another consequence of Lemma 2.2 is the following Proposition. For $\theta \in S^{n-1}$, we define the $\psi_2$-norm of the linear functional $\langle \cdot, \theta \rangle$ with respect to the uniform measure on $K$ as:

$$\|\langle \cdot, \theta \rangle\|_{L_{\psi_2}(K)} := \inf \left\{ \lambda > 0; \frac{1}{|K|} \int_K e^{\lambda \langle x, \theta \rangle^2} dx \leq 2 \right\}.$$

The $L_p$-norm is defined as:

$$\|\langle \cdot, \theta \rangle\|_{L_p(K)} := \left( \frac{1}{|K|} \int_K |\langle x, \theta \rangle|^p dx \right)^{1/p}.$$

It is well-known (e.g. [17, Proposition 3.6]) that:

$$\|\langle \cdot, \theta \rangle\|_{L_{\psi_2}(K)} \simeq \sup_{p \geq 2} \frac{\|\langle \cdot, \theta \rangle\|_{L_p(K)}}{\sqrt{p}},$$

implying in particular that:

$$\|\langle \cdot, \theta \rangle\|_{L_{\psi_2}(K)} \geq C_1 \frac{\|\theta\|_K^*}{\sqrt{n}},$$

(2.5)
since \( \|\theta\|_K^* \simeq \|\langle \cdot, \theta \rangle\|_{L_n(K)} \) (e.g. [32]). By Lemma 2.2, we readily see that:

\[
\frac{1}{|K|} \int_K e^{\langle x, \theta \rangle^2 / \lambda^2} dx = 1 + \frac{1}{|K|} \int_1^\infty \text{Vol} \left\{ x \in K; e^{\langle x, \theta \rangle^2 / \lambda^2} > t \right\} dt \leq 1 + \int_1^\infty e^{-\alpha n (\|\theta\|_K^*)^2 \log t} dt,
\]

so choosing \( \lambda = C_2 \|\theta\|_K^* \sqrt{\alpha n} \) for an appropriate value of \( C_2 > 0 \), the latter expression is smaller than 2. We conclude:

**Proposition 2.4.** Let \( K \subset \mathbb{R}^n \) be a centrally-symmetric 2-convex body with constant \( \alpha \). Then for all \( \theta \in S^{n-1} \):

\[
C_1 \|\theta\|_K^* \sqrt{\frac{n}{\alpha}} \leq \|\langle \cdot, \theta \rangle\|_{L\psi_2(K)} \leq C_2 \|\theta\|_K^* \sqrt{\frac{\alpha}{n}},
\]

where \( C_1, C_2 > 0 \) are two universal constants.

Proposition 2.4 provides us with a way to find directions \( \theta \in S^{n-1} \) for which \( \text{Vol} \left\{ x \in K; \langle x, \theta \rangle \geq t \right\} \) decays in a sub-gaussian rate, as reflected by \( \|\langle \cdot, \theta \rangle\|_{L\psi_2(K)} \). As a first application, note that for any convex body of volume one, there exists a direction in which the width is smaller than \( \sqrt{n} \) (otherwise the body would contain a Euclidean ball of volume greater than one). Together with a straightforward application of Markov’s inequality, and denoting \( M^*(K) = \int_{S^{n-1}} \|\theta\|_K^* d\sigma(\theta) \), we conclude the following immediate corollary of Proposition 2.4.

**Corollary 2.5.** Let \( K \subset \mathbb{R}^n \) be a centrally-symmetric convex body. Assume that \( K \) is 2-convex with constant \( \alpha \) and volume 1. Then there exists a universal constant \( C > 0 \) such that:

1. There exists a \( \theta \in S^{n-1} \) such that:

\[
\|\langle \cdot, \theta \rangle\|_{L\psi_2(K)} \leq C / \sqrt{\alpha}.
\]

2. \( \sigma \left\{ \theta \in S^{n-1}; \|\langle \cdot, \theta \rangle\|_{L\psi_2(K)} \leq C \frac{M^*(K)}{\sqrt{\alpha} \sqrt{n}} \right\} \geq \frac{1}{2} .
\]

In Section 4, we will see several positions of a 2-convex body \( K \) of volume 1 for which \( M^*(K) \leq C \sqrt{n} \). The last corollary implies that in these positions, at least half of the directions have \( \psi_2 \)-decay. We say that a body satisfying:

\[
\|\langle \cdot, \theta \rangle\|_{L\psi_2(K)} \leq A \cdot |K|^{1/n}
\]

for all \( \theta \in S^{n-1} \) is a \( \psi_2 \) body (with constant \( A \)). In general, a 2-convex body is not a \( \psi_2 \) body. Indeed, as apparent from (2.5), a \( \psi_2 \) body (with constant \( A \)) of volume 1 always satisfies \( \text{diam}(K) \leq CA \sqrt{n} \), but any \( l_p^n \) for \( p < 2 \) (normalized to have volume 1) already fails to satisfy this (with a universal constant \( A \)) for large enough \( n \). Here and henceforth, \( \text{diam}(K) \) denotes the diameter of \( K \). Nevertheless, we can still say the following:

**Proposition 2.6.** Let \( K \subset \mathbb{R}^n \) be a centrally-symmetric convex body. Assume that \( K \) is 2-convex with constant \( \alpha \), has volume 1 and that it is isotropic. Then a random \( \lfloor n/2 \rfloor \)-dimensional section of \( K \) is a \( \psi_2 \)-body with high probability.
Proof. By definition, any section of $K$ is a 2-convex body with the same constant. By Proposition 1.3, the isotropic position is also a finite volume-ratio position for $K$, and $c\sqrt{\alpha}\sqrt{n}D_n \subset K$. But by a classical result of [37] and [38] (based on [19]), a random $\lfloor n/2 \rfloor$-dimensional section $L \cap E$ of a convex body $L$ containing $D_n$ is isomorphic to a Euclidean ball, and in particular satisfies $\text{diam}(L \cap E) \leq C(|L|/|D_n|)^{2/n}$ with probability greater than $1 - (1/2)^n$. Therefore:

\[
(2.6) \quad c\sqrt{\alpha}\sqrt{n}(D_n \cap E) \subset K \cap E \subset \frac{C'}{\sqrt{\alpha}}\sqrt{n}(D_n \cap E)
\]

with the same probability. Applying Proposition 2.4 to $K \cap E$ and using the left-hand-side of (2.6) to compensate for the volume of $K \cap E$, we see that:

\[
\|\langle \cdot, \theta \rangle\|_{L^2(K \cap E)} \leq \frac{C'}{\alpha^{3/2}} |K \cap E|^{2/n}
\]

for all $\theta \in S^{n-1} \cap E$. This concludes the proof. □

3. Operations preserving 2-convexity

We have already seen that (by definition) any section of a 2-convex body with constant $\alpha$ is itself a 2-convex body with the same constant. In this section we will consider several additional natural operations which preserve 2-convexity and the dual notion of 2-smoothness, and conclude with several new results on the isotropic constant of different families of bodies.

The first natural operation to consider is taking projections. Since this is the dual operation to taking sections, it will be convenient to first introduce the dual notion to 2-convexity, which is 2-smoothness. The modulus of smoothness of $K$ is defined as the following function for $\tau > 0$:

\[
\rho_K(\tau) = \sup \left\{ \frac{\|x + y\|_K + \|x - y\|_K}{2} - 1 : \|x\|_K \leq 1, \|y\|_K \leq \tau \right\}.
\]

A body $K$ is called “2-smooth with constant $\beta$” (see, e.g. [24, Chapter 1.e]), if for all $\tau > 0$:

\[
(3.2) \quad \rho_K(\tau) \leq \beta\tau^2.
\]

It is well-known (e.g. [24]) that the modulus of smoothness is dual to the modulus of convexity (this can be carefully formalized using Legendre transforms). We summarize Propositions 1.e.2 and 1.e.6 from [24] in the following:

Lemma 3.1. Let $K$ be a centrally-symmetric convex body in $\mathbb{R}^n$. Then $K$ is 2-convex with constant $\alpha$ iff $K^\circ$ is 2-smooth with constant $1/\alpha$.  

We will frequently refer to the Blaschke-Santalo inequality ([35], the right hand side below) and its reverse form due to Bourgain-Milman ([7], the left hand side below), which together state that for any convex body $K$:

\[
c \leq \left( \frac{|K|}{|D_n|} \right)^{1/n} \left( \frac{|K^\circ|}{|D_n|} \right)^{1/n} \leq 1.
\]
Lemma 3.1, coupled with the Blaschke-Santalo inequality or its reverse form, imply that we can translate many volumetric results on 2-convex bodies to 2-smooth bodies. In particular, Proposition 1.3 translates to the fact that 2-smooth bodies have finite outer-volume-ratio. We define the outer-volume-ratio of a body $K$ as:

$$
o.v.r.(K) = \min_{E \supseteq K} \left( \frac{|E|}{|K|} \right)^{\frac{1}{n}}$$

where the minimum is taken over all ellipsoids that contain $K$. If $o.v.r.(K) < C$, for some universal constant $C > 0$, it is customary to say that $K$ has finite outer-volume-ratio.

It is well known (e.g. [29]) that $L_K \leq C' o.v.r.(K)$ for any convex body $K$. Combining everything together, we have the following useful:

**Proposition 3.2.** Let $K$ be a 2-smooth convex body with constant $\beta$. Then $o.v.r.(K) \leq C \sqrt{\beta}$. In particular, $L_K \leq C' \sqrt{\beta}$.

Note that if $K \subset T$ then $o.v.r.(K) \leq (|T|/|K|)^{1/n} o.v.r.(T)$. The following is therefore an immediate corollary of Proposition 3.2:

**Corollary 3.3.** Let $K$ be a centrally-symmetric convex body in $\mathbb{R}^n$. Then:

$$L_K \leq C \inf \left\{ \sqrt{\beta} \left( \frac{|T|}{|K|} \right)^{1/n} \bigg| K \subset T, \ T \text{ is } 2\text{-smooth with constant } \beta \right\}$$

We can now turn to investigate the action of taking projections of 2-convex and 2-smooth bodies. For a subspace $E \subset \mathbb{R}^n$, we denote by $\text{Proj}_E$ the orthogonal projection onto $E$. As evident from the definitions, any section of a 2-smooth body with constant $\beta$ is itself a 2-smooth body with the same constant. By passing to the polar body and using Lemma 3.1, the duality between sections and projections immediately implies:

**Lemma 3.4.** Let $K \subset \mathbb{R}^n$ be a 2-convex (2-smooth) body with constant $\gamma$. Then so is $\text{Proj}_E(K)$, with the same constant $\gamma$, for any subspace $E \subset \mathbb{R}^n$.

Using Lemma 3.4, a remarkable consequence of Proposition 2.4 is that the $\psi_2$-norm of the linear functional $\langle \cdot, x \rangle$ on a projection $\text{Proj}_E(K)$ of a 2-convex body $K$, essentially depends (up to universal constants) only on $x \in E$ and not on the subspace $E$. More precisely:

**Proposition 3.5.** Let $K \subset \mathbb{R}^n$ be a 2-convex body with constant $\alpha$, and let $E$ be a $k$-dimensional subspace. Then for any $x \in E$:

$$C_1 \|x\|_{K}^* \leq \|\langle \cdot, x \rangle\|_{L_{\psi_2}(\text{Proj}_E(K))} \sqrt{k} \leq C_2 \frac{1}{\sqrt{\alpha}} \|x\|_{K}^*$$

This is one of the rare cases where we can deduce volumetric information on $\text{Proj}_E(K)$ from that of $K$. Typically, these two bodies have different volumetric behaviour.

Let us consider other natural operations which preserve 2-convexity. Unfortunately, the Minkowski sum is a bad candidate for this. Indeed, even in $\mathbb{R}^2$, the sum of two very narrow ellipsoids which are perpendicular to each other, may be brought arbitrarily close to a square, which is not 2-uniformly convex. Nevertheless, there exists a well known natural summation operation, which actually preserves both 2-uniform convexity and 2-uniform...
smoothness. Recall that the 2-Firey sum of two convex bodies $K$ and $T$, denoted by $K +_2 T$, is defined as the unit ball of the norm satisfying:

$$\|z\|_{K+_2 T}^2 = \inf_{z = x + y} \|x\|_K^2 + \|y\|_T^2.$$ 

It is easy to see that the dual norms satisfy:

$$(\|z\|_{K+_2 T}^*)^2 = (\|z\|_K^*)^2 + (\|z\|_T^*)^2.$$ 

We will refer to the latter operation as 2-Firey intersection, and denote the 2-Firey intersection of $K$ and $T$ as $K \cap_2 T$. Note that $(K \cap_2 T)^o = K^o +_2 T^o$.

**Lemma 3.6.** Let $K$ and $T$ be 2-convex (smooth) bodies with constants $\gamma_K$ and $\gamma_T$, respectively. Then so is their 2-Firey sum $K +_2 T$ and intersection $K \cap_2 T$, with constant $\min\{\gamma_K, \gamma_T\}/8 (\max\{\gamma_K, \gamma_T\} \cdot 8)$.

**Proof.** Obviously there is no loss in generality in assuming that $\gamma_K = \gamma_T = \gamma$. Since $(K \cap_2 T)^o = K^o +_2 T^o$, Lemma 3.1 implies that the case of 2-smooth bodies follows from the case of 2-convex bodies by duality. We will therefore restrict ourselves to the latter case, and assume that $K$ and $T$ are 2-convex with constant $\gamma$.

By Lemma 2.1, we have for $G = K, T$ and for all $x, y \in \mathbb{R}^n$:

$$\|x\|_G^2 + \|y\|_G^2 - 2\|\frac{x + y}{2}\|_G^2 \geq \frac{\gamma}{2} \|x - y\|_G^2. \tag{3.3}$$

Summing these two inequalities for $G = K$ and $G = T$, we see that (3.3) is also satisfied for $G = K \cap_2 T$. Using Lemma 2.1 again, this implies that $K \cap_2 T$ is 2-convex with constant $\gamma/8$.

Next, for any $z_1, z_2 \in \mathbb{R}^n$, write $z_i = x_i^K + x_i^T$ so that:

$$\|z_i\|_{K+_2 T}^2 = \|x_i^K\|_K^2 + \|x_i^T\|_T^2$$

(by compactness the infimum is achieved). By Lemma 2.1, we know that for $G = K, T$:

$$\|x_i^K\|_G^2 + \|x_i^T\|_G^2 \geq 2\left(\|\frac{x_i^K + x_i^T}{2}\|_G^2 + \frac{\gamma}{2} \|x_i^K - x_i^T\|_G^2\right).$$

Summing these two inequalities for $G = K$ and $G = T$ and denoting $Z = K +_2 T$, we have:

$$\|z_1\|_Z^2 + \|z_2\|_Z^2 = \|x_1^K\|_K^2 + \|x_2^K\|_K^2 + \|x_1^T\|_T^2 + \|x_2^T\|_T^2$$

$$\geq 2\left(\left\|\frac{x_1^K + x_2^K}{2}\right\|_K^2 + \left\|\frac{x_1^T + x_2^T}{2}\right\|_T^2\right) + \frac{\gamma}{2} \left(\|x_1^K - x_2^K\|_K^2 + \|x_1^T - x_2^T\|_T^2\right)$$

$$\geq \frac{\gamma}{2} \left\|\frac{z_1 + z_2}{2}\right\|_Z^2 + \frac{\gamma}{2} \|z_1 - z_2\|_Z^2,$$

where the last inequality follows from the definition of $Z = K +_2 T$ and the fact that $z_1 + z_2 = (x_1^K + x_2^K) + (x_1^T + x_2^T)$ and $z_1 - z_2 = (x_1^K - x_2^K) + (x_1^T - x_2^T)$. Lemma 2.1 implies that $K +_2 T$ is 2-convex with constant $\gamma/8$. 

**Remark 3.1.** It is important to emphasize that the additional factor of 8 appearing in the Lemma is immaterial, and that the Lemma holds in full generality when summing (intersecting) an arbitrary number of bodies (with the same constant factor of 8).
We can now summarize our bounds for the isotropic constant in the following statements. For a Banach space \( X \), we denote by \( SQ_n(X) \) the class of unit balls of \( n \)-dimensional subspaces of quotients of \( X \). We denote \( F_0^2SQ_n(X) = SQ_n(X) \), and by induction:

\[
F_{i+1}^2SQ_n(X) = \left\{ \bigwedge_{i=1}^l \bigoplus_{j=1}^{m_i} K_{i,j} ; \{ K_{i,j} \} \subset F_i^2SQ_n(X) \right\},
\]

where \( \bigwedge \) and \( \bigoplus \) denote 2-Firey intersection and sum, respectively. We set \( F_2^2SQ_n(X) = \bigcup_{i=0}^{\infty} F_i^2SQ_n(X) \). Note that it is possible to make the class \( F_2^2SQ_n(X) \) even richer, by alternately taking subspaces, quotients, 2-Firey sums and 2-Firey intersections (since the operation of 2-Firey sum is not distributive with respect to taking subspace or 2-Firey intersection) starting from \( X \), but this is a complication which we wish to avoid. Lemmas 3.4 and 3.6, together with Remark 3.1, show that if \( X \) is 2-convex (2-smooth) with constant \( \alpha (\beta) \), then so is every member of \( F_2^2SQ_n(X) \) with constant \( \alpha/8 (8\beta) \). Corollary 3.3 therefore implies:

**Theorem 3.7.** Let \( K \) be a centrally-symmetric convex body in \( \mathbb{R}^n \), and let \( X \) be a 2-smooth Banach space with constant \( \beta \). Then:

\[
L_K \leq C \sqrt{\beta} \inf \left\{ \left( \frac{|T|}{|K|} \right)^{1/n} \bigg| K \subset T, T \in F_2^2SQ_n(X) \right\}.
\]

Consider \( X = L_p \) for \( 2 \leq p < \infty \) in Theorem 3.7. Note that \( X^* = L_q \) with \( q = 1 + 1/(p - 1) \), for which it is known (e.g. [24, p. 63]) that \( X^* \) is 2-convex with constant equivalent to \( 1/(p - 1) \). By Lemma 3.1 this implies that \( X \) is 2-smooth with constant bounded by \( C(p - 1) \). We therefore have:

**Corollary 3.8.** Let \( K \) be a centrally-symmetric convex body in \( \mathbb{R}^n \). Then:

\[
L_K \leq C \inf \left\{ \sqrt{p} \left( \frac{|T|}{|K|} \right)^{1/n} \bigg| K \subset T, T \in F_2^2SQ_n(L_p) \right\}.
\]

This is a generalization of one half (the range \( p \geq 2 \)) of a Theorem of Junge ([18], see also [27]):

**Theorem (Junge).**

\[
L_K \leq C \inf \left\{ \sqrt{q} \left( \frac{|T|}{|K|} \right)^{1/n} \bigg| K \subset T, T \in SQ_n(L_p) \right\}.
\]

In fact, Junge showed that \( L_p \) may be replaced by any Banach space \( X \) with finite type and bounded \( gl_2(X) \) (the Gordon-Lewis constant of \( X \)), in which case \( \sqrt{p} q \) above should be replaced by some constant depending on \( X \).

We can also improve the second half of Junge’s Theorem (in the range \( 1 < p \leq 2 \)) by replacing the factor of \( q \) by \( \sqrt{q} \). Unfortunately, with our approach we have to insist that \( K \) itself is in \( F_2^2SQ_n(L_p) \). Our version reads as follows:

**Theorem 3.9.** Let \( K \in F_2^2SQ_n(L_p) \) for \( 1 < p \leq 2 \), and let \( q \) be given by \( 1/p + 1/q = 1 \). Then:

\[
L_K \leq C \sqrt{q}.
\]
The latter is an immediate corollary of the the fact that $L_p$ for $1 < p \leq 2$ is 2-convex with constant equivalent to $p - 1$ (e.g. [24, Chapter 1.e]), combined with the following general Theorem, which is a consequence of Proposition 1.1:

**Theorem 3.10.** Let $X$ be a 2-convex Banach space with constant $\alpha$, and let $K \in F_2^{SQ_n}(X)$. Then:

$$L_K \leq C \frac{1}{\sqrt{\alpha}}.$$

Another interesting example is obtained by taking $X$ to be the space of all $m \times m$ complex or real matrices, equipped with the norm $\|A\| = (\text{tr}(AA^*)^{p/2})^{1/p}$, the so-called $l_p$-Schatten-Class which will be denoted by $S_p^m$. It was observed in [22] that the isotropic constants of these spaces are uniformly bounded (in $m$), which is especially interesting in the range $1 \leq p < 2$, since for $p \geq 2$ it is known that the unit ball of $S_p^m$ (or any of its subspaces) has finite outer volume-ratio. In the former range, it has been recently shown in [14] that (in particular) the isotropic constants of several special subspaces of $S_p^m$ are also uniformly bounded. Although our method does not extend to $p = 1$, we can show the following result, which in particular demonstrates that the same is true for any subspace of quotient of $S_p^m$, provided that $p$ is bounded away from 1. The modulus of convexity (and smoothness) of $S_p^m$ was estimated by N. Tomczak-Jaegermann in [39], where it was shown that $\delta_{S_p^m} \simeq \delta_{L_p}$. It follows that $S_p^m$ is 2-convex with constant equivalent to $p - 1$ for $1 < p \leq 2$, which together with Theorem 3.10 gives:

**Theorem 3.11.** Let $K \in F_2^{SQ_n}(S_p^m)$ for $1 < p \leq 2$ and $m \geq n$, and let $q$ be given by $1/p + 1/q = 1$. Then:

$$L_K \leq C \sqrt{q}.$$

It is clear that the case $p = 1$ in Theorem 3.9 and Theorem 3.11 must serve as a breakdown point for our method. Indeed, since $S_p^m$ contains $l_1^m$ as a subspace (of the diagonal matrices), and since every convex body may be approximated as the unit ball of a quotient of $l_1^m$ for large-enough $m$, or simply as the quotient of $L_1$, a similar result for $p = 1$ in either theorem would solve the Slicing Problem.

### 4. Equivalence between positions of 2-convex bodies

For the results of this section, we will need to recall a few basic notions from Banach space theory. The (Rademacher) type-$p$ constant of a Banach space $X$ (for $1 \leq p \leq 2$), denoted $T_p(X)$, is the minimal $T > 0$ for which:

$$\left( \mathbb{E} \left\| \sum_{i=1}^{m} \varepsilon_i x_i \right\|^2 \right)^{1/2} \leq T \left( \sum_{i=1}^{m} \|x_i\|^p \right)^{1/p}$$

for any $m \geq 1$ and any $x_1, \ldots, x_m \in X$, where $\{\varepsilon_i\}$ are independent, symmetric $\pm 1$-valued random variables and $\mathbb{E}$ denotes expectation. Similarly, the cotype-$q$ constant of $X$ (for $2 \leq q \leq \infty$), denoted $C_q(X)$, is the minimal $C > 0$ for which:

$$\left( \mathbb{E} \left\| \sum_{i=1}^{m} \varepsilon_i x_i \right\|^2 \right)^{1/2} \geq \frac{1}{C} \left( \sum_{i=1}^{m} \|x_i\|^q \right)^{1/q}$$
for any \( m \geq 1 \) and \( x_1, \ldots, x_m \in X \). We say that \( X \) has type \( p \) (cotype \( q \)) if \( T_p(X) < \infty \) (\( C_q(X) < \infty \)). We also say that \( X \) is of type \( p \) (cotype \( q \)) if \( p = \sup \{ p' : X \) has type \( p' \} \) \((q = \inf \{ q' : X \) has cotype \( q' \})

Let \( L_2(\{-1, 1\}^m, X) \) denote the space of \( X \)-valued functions on the discrete cube \( \{-1, 1\}^m \), equipped with the norm \((\mathbb{E}\|f(\varepsilon_1, \ldots, \varepsilon_m)\|^2)^{1/2} \). We denote by \( \text{Rad}_m(X) \) the Rademacher projection on \( L_2(\{-1, 1\}^m, X) \) (see [30]), and denote \( \|\text{Rad}(X)\| = \sup_m \|\text{Rad}_m(X)\| \) where \( \|\text{Rad}_m(X)\| \) is the operator norm of \( \text{Rad}_m(X) \). By duality, it is easy to verify that \( \|\text{Rad}(X^*)\| = \|\text{Rad}(X)\| \), and it is clear that \( \|\text{Rad}_m(X)\| = \sup_{E \subset X} \|\text{Rad}_m(E)\| \) where the supremum runs over all finite-dimensional subspaces of \( X \).

One of the most important results in the local-theory of Banach spaces is a theorem by Pisier who showed that \( \|\text{Rad}(X)\| \) may be bounded from above by an (explicit) function of \( T_2(X) \) when \( p > 1 \), concluding that \( \|\text{Rad}(X)\| < \infty \) when \( X \) has type \( p \) \( > 1 \). When \( p = 2 \), there is a much easier argument, going back to a remark at the end of the work by Maurey and Pisier [26] (see also [4, Remark 2.11] for an explicit proof), showing (without any constants!):

**Lemma 4.1.** \( \|\text{Rad}(X)\| \leq T_2(X) \).

The next lemma, which gives a non-quantitative estimate of the opposite inequality (for the general \( p \) case) using a compactness argument, is a known consequence of the Maurey-Pisier Theorem [26]. We have not been able to find a reference for it, so we sketch the proof below.

**Lemma 4.2.** There exists a function \( C(R) : \mathbb{R}_+ \to \mathbb{R}_+ \) such that any finite-dimensional Banach space \( X \) with \( \|\text{Rad}(X)\| \leq R \) satisfies \( T_p(X) \leq C(R) \) with \( p(R) = 1 + 1/C(R) \).

**Sketch of proof.** Assume that this is not true for some \( R > 0 \). This means that there exist finite-dimensional Banach spaces \( X_i \) with \( \|\text{Rad}(X_i)\| \leq R \) and \( T_{1+1/i}(X_i) > i \). The latter easily implies that \( \dim(X_i) \to \infty \), since always \( T_p(X_i) \leq T_2(X_i) \leq \sqrt{\dim(X_i)} \) for any \( 1 \leq p \leq 2 \) \((X_i \cong \sqrt{\dim(X_i)}\)-isomorphic to a Hilbert space \( H_i \) by John’s Theorem, and \( T_2(H_i) = 1 \)). We now construct an infinite dimensional Banach space \( X \) as the \( l_2 \) sum of the \( X_i \)'s, i.e. for \( x = (x_i)_{i \geq 1} \) with \( x_i \in X_i \) define \( \|x\|_X = (\sum_{i \geq 1} \|x_i\|_{X_i}^2)^{1/2} \) and set \( X = \{x : \|x\|_X < \infty \} \) endowed with the norm \( \|\cdot\|_X \). It is elementary to check that \( \|\text{Rad}(X)\| \leq R \), and since \( X \) contains each \( X_i \) as a subspace we must have that \( X \) is of type \( 1 \). The latter implies by the Maurey-Pisier Theorem (actually we only need the type \( 1 \) case, which is due to Pisier [33]) that \( X \) contains \((1+\epsilon)\) isometric copies of \( l^m_1 \) for arbitrary \( \epsilon > 0 \) and \( m \), and as a consequence \( \|\text{Rad}(X)\| \geq \sup_m \|\text{Rad}(l^m_1)\| = \infty \). We arrive to a contradiction, so the assertion is proved. Noted that the choice of \( p(R) \) as a function of \( C(R) \) was arbitrary, and any function \( p(R) \) decreasing to \( 1 \) as \( C(R) \) tends to infinity works equally well.

Let us return to the study of 2-convex bodies. We recall the following classical result (e.g. [24, Theorem 1.e.16]). For completeness, we sketch the proof.

**Lemma 4.3.**

1. Let \( K \) be a 2-convex body with constant \( \alpha \). Then \( C_2(X_K) \leq \frac{C}{\alpha} \).
2. Let \( K \) be a 2-smooth body with constant \( \beta \). Then \( T_2(X_K) \leq C\sqrt{\beta} \).
Proof. (1) easily follows from the equivalent characterization (2.1) of a 2-convex body, which asserts that for any \( x_1, x_2 \in \mathbb{R}^n \):
\[
\mathbb{E} \| \varepsilon_1 x_1 + x_2 \|^2 = \frac{1}{2} (\| x_2 + x_1 \|^2 + \| x_2 - x_1 \|^2) \geq \alpha \| x_1 \|^2 + \| x_2 \|^2.
\]
Hence by induction, since \( \alpha < 1 \):
\[
\mathbb{E} \left\| \sum_{i=1}^{m} \varepsilon_i x_i \right\|^2 \geq \alpha \sum_{i=1}^{m} \| x_i \|^2,
\]
for any \( x_1, \ldots, x_m \in \mathbb{R}^m \), which concludes the proof of (1) (even without a constant!). (2) follows either by duality or similarly from the equivalent characterization of a 2-smooth body (e.g. [3, Theorem A.7]):
\[
\| x + y \|^2 + \| x - y \|^2 - 2 \| x \|^2 \leq C \beta \| y \|^2,
\]
for every \( x, y \in \mathbb{R}^n \). \( \square \)

We are now ready to conclude the following useful:

**Lemma 4.4.** Let \( K \) be a 2-convex body with constant \( \alpha \). Then:

(1) \[ \| \text{Rad}(X_K) \| \leq C / \sqrt{\alpha}. \]

(2) There exists a \( p > 1 \) which depends on \( \alpha \) only, such that:
\[ T_p(X_K) \leq 1 / (p - 1). \]

**Proof.** By Lemma 3.1, \( K^\circ \) is 2-smooth with constant \( 1 / (16 \alpha) \), and so by Lemmas 4.1 and 4.3 we see that:
\[ \| \text{Rad}(X) \| = \| \text{Rad}(X^*) \| \leq T_2(X^*) \leq \frac{C}{\sqrt{\alpha}}, \]
which concludes the proof of (1). Applying Lemma 4.2, we immediately deduce (2). \( \square \)

Lemmas 4.3 and 4.4 allow us to deduce several interesting results about 2-convex bodies. By a classical result of Figiel and Tomczak-Jaegermann on the \( l\)-position ([10]), for any convex body \( K \) there exists a position for which \( M(K) M^*(K) \leq C \| \text{Rad}(X_K) \| \), and in fact this is satisfied in the minimal mean-width position. The latter is defined (up to orthogonal rotations) as the volume-preserving affine image of \( K \) for which \( M^*(K) \) is minimal. Recall that we always have:

\[ \frac{1}{M(K)} \leq \text{Vol.rad.}(K) \leq M^*(K), \]

where \( \text{Vol.rad.}(K) = (|K| / |D_n|)^{1/n} \) and the first inequality follows from Jensen’s inequality while the second is Urysohn’s inequality. We therefore deduce that in the minimal mean-width position, a 2-convex body \( K \) with constant \( \alpha \) satisfies:

\[ M^*(K) \leq \frac{C}{\sqrt{\alpha}} \text{Vol.rad.}(K), \]

\[ M(K) \leq \frac{C}{\sqrt{\alpha}} \text{Vol.rad.}(K)^{-1}, \]
which are essentially the best possible by (4.1). We will refer to (4.2) as "$M^*(K)$ is bounded", omitting the reference to the volume-radius. As we shall see, there are many advantages of working with a position in which $M^*(K)$ is bounded.

Our next Proposition shows that in the case of 2-convex bodies, $K$ must be essentially isotropic whenever we have a good upper bound on $M^*(K)$. For convenience, we define $M^*_2(K) = (\int_{S^{n-1}}(\|\theta\|^2_K)^2 d\sigma(\theta))^{1/2}$, which is well known to be equivalent to $M^*(K)$ (by Kahane’s inequality for instance).

**Proposition 4.5.** For any 2-convex body $K$ with constant $\alpha$ and volume 1, we have:

$$\int_K |x| \, dx \leq C \frac{M^*(K)}{\sqrt{\alpha}}.$$

**Proof.**

$$\int_K |x| \, dx \leq \left( \int_K |x|^2 \, dx \right)^{1/2} = \sqrt{n} \left( \int_{S^{n-1}} \langle x, \theta \rangle^2 \, d\sigma(\theta) \, dx \right)^{1/2}$$

$$= \sqrt{n} \left( \int_{S^{n-1}} \langle x, \theta \rangle^2 \, d\sigma(\theta) \right)^{1/2} \leq \frac{C'}{\sqrt{\alpha}} \left( \int_{S^{n-1}} (\|\theta\|_K^4)^2 d\sigma(\theta) \right)^{1/2},$$

where we used Proposition 2.4 in the last inequality. The last term is equal to $\frac{C''}{\sqrt{\alpha}} M^*_2(K)$, which is majorized by $\frac{C''}{\sqrt{\alpha}} M^*(K)$. $\square$

The last Proposition has an interesting consequence regarding 2-Firey sums of 2-convex bodies in minimal mean-width position, or in any bounded $M^*$ position in general.

**Corollary 4.6.** Let $K$ and $T$ be 2-uniformly convex bodies, such that $M^*_2(K) \leq C_K \text{Vol.rad.}(K)$ and $M^*_2(T) \leq C_T \text{Vol.rad.}(T)$ (and therefore essentially isotropic). Then $M^*_2(K +_T T) \leq \max(C_K, C_T) \text{Vol.rad.}(K +_T T)$. In particular, $K +_T T$ is essentially isotropic.

**Proof.** Notice that $(M^*_2)^2$ is clearly additive with respect to 2-Firey sums, whereas by [25] $|K +_T T|^{2/n} \geq |K|^{2/n} + |T|^{2/n}$. The claim then easily follows. $\square$

An additional property of any position for which $M^*(K)$ is bounded, is that it automatically satisfies half of the conditions of being in a 2-regular M-position. Recall that a convex body $K$ in $\mathbb{R}^n$ is said to be in $a$-regular M-position ($0 < a \leq 2$) if its homothetic copy $K'$, normalized so that $|K'| = |D_n|$, satisfies:

\begin{equation}
N(K', tD_n) \leq \exp(Cn/t^a) \quad \text{and} \quad N((K')^c, tD_n) \leq \exp(Cn/t^a),
\end{equation}

for $t \geq 1$, where $N(K, L)$ is the covering number of $K$ by $L$ (see [12]) and $C > 0$ is a universal constant. It was shown by Pisier ([34]) that an $a$-regular M-position for $0 < a < 2$ always exists (with a constant $C$ in (4.4) depending only on $a$). When $M^*(K)$ is bounded and $|K| = |D_n|$, by Sudakov’s inequality ([12]):

$$N(K, tD_n) \leq \exp(Cn(M^*(K)/t)^2) \leq \exp(Cn/t^2)$$
for $t \geq 1$, so half of the condition for being in a 2-regular M-position is satisfied. In general, the other half of the condition, namely:

\[(4.5) \quad N(K^o, tD_n) \leq \exp(Cn/t^2),\]

does not follow from knowing that $M^*(K)$ is bounded. Nevertheless, we mention two cases where this would follow. If $K$ is in minimal mean-width position and $|K| = |D_n|$, in which case both $M^*(K)$ and $M(K)$ are bounded using (4.2) and (4.3), then (4.5) follows from Sudakov’s inequality applied to $K^o$. Another case is when $K$ is in a finite volume-ratio position with bounded $M^*(K)$ (remember that we know that $K$ has finite volume-ratio), in which case (4.5) is trivially satisfied. The second case, if it exists, will be preferred over the first, since it adds the finite-volume ratio position property (which is not guaranteed in general by the minimal mean-width position), in particular implying that $M(K)$ is bounded.

Luckily, for a 2-convex body, there exists an “all-in-one” position which gives all of the above mentioned properties: bounded $M^*$, having finite volume-ratio (and therefore being in a 2-regular M-position) and essential isotropicity. This position is exactly John’s maximal-volume ellipsoid position. This follows from the following useful lemma from [27] (which appeared first in an equivalent form in [9]):

**Lemma 4.7.** For any convex body $K$ in John’s maximal-volume ellipsoid position, the following holds:

\[ M^2_2(K)b(K) \leq T_2(X_K*), \]

where $b(K) = \max_{\theta \in S^{n-1}} \|\theta\|_K$.

For a 2-convex body $K$ with constant $\alpha$, the polar body is 2-smooth with constant $1/(16\alpha)$, and therefore by Lemma 4.3, $X_K^*$ has type 2 with constant $T_2(X_K^*) \leq C/\sqrt{\alpha}$. Noting that $M^*(K) \leq M^*_2(K)$, Lemma 4.7 therefore gives:

**Corollary 4.8.** A 2-convex body $K$ with constant $\alpha$ in John’s maximal-volume ellipsoid position, satisfies:

\[ M^*(K)b(K) \leq C/\sqrt{\alpha}. \]

Since $M^*(K)b(K)$ is invariant under homothety, we may assume above that $|K| = |D_n|$, in which case $b(K) \geq 1$ (by volume consideration) and $M^*(K) \geq 1$ (by Urysohn’s inequality). We therefore see that in John’s maximal-volume ellipsoid position $M^*(K) \leq C/\sqrt{\alpha} \text{Vol.rad.}(K)$. The similar bound on $b$ implies again that $K$ has finite-volume ratio, $v.r.(K) \leq C/\sqrt{\alpha}$, with the same bound (up to a possible constant) as in Proposition 1.3. Proposition 4.5 coupled with the latter bound on $M^*(K)$ in John’s position, imply Proposition 1.4 stated in the Introduction.

One last additional property that we would like our ”all-in-one” position to satisfy is having a small-diameter: if $|K| = |D_n|$, we would like to have $\text{diam}(K) \leq C(n/\log n)^{1/2}$. The motivation for this requirement comes from [1], where it was shown that if an isotropic 2-convex body has small-diameter in the above sense, then most of its marginals are approximately Gaussian (see [1] or Section 5 for more details). It is easy to check that this requirement is indeed satisfied by all the $l^p_1$ unit balls for $1 < p \leq 2$ (normalized to have the appropriate volume).
Unfortunately, the small-diameter requirement is not satisfied for a general 2-convex body in isotropic position, as illustrated by the following:

**Example 4.9.** Let:

\[ T = \left\{ (x, y) \in \mathbb{R}^2 : x^2 + (|y| + 1)^2 \leq 2 \right\}. \]

The set \( T \) is 2-convex with constant \( c \), and has two “cusps”, at \((1, 0)\) and \((-1, 0)\). Denote by \( K \subset \mathbb{R}^n \) the revolution body of \( T \) around the \( y \)-axis, namely:

\[ K = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \left( x_1^2 + \ldots + x_{n-1}^2, x_n \right) \in T \right\}. \]

It is easy to check that \( K \) is 2-convex with constant \( c' \). Let \( \tilde{K} \subset \mathbb{R}^n \) be an isotropic image of \( K \) of volume 1. Then \( \text{diam}(\tilde{K}) \geq c'n \).

**Sketch of proof.** Around its “cusp” hyperplane \( e_n^\perp \), \( K \) looks like a two-sided cone, and therefore half of the volume of \( K \) lies inside the slab \( \{ x \in \mathbb{R}^n : |\langle x, e_n \rangle| \leq c(n)/n \} \) with \( c(n) \approx 1 \). But in isotropic position of volume 1, half of the volume of \( \tilde{K} \) lies inside slabs of width in the order of \( L_K \) (and \( L_K \approx 1 \) by Proposition 1.1). This means that we must inflate \( K \) by an order of \( n \) in the direction of \( e_n \) when passing to \( \tilde{K} \), implying that \( \text{diam}(\tilde{K}) \geq c'n \). \( \square \)

Nevertheless, the following proposition shows that in L"owner’s minimal-volume outer ellipsoid position, the small-diameter requirement is satisfied, although we are not able to guarantee any of the other “good” properties satisfied by John’s maximal-volume ellipsoid position. We note that \( K \) is in L"owner’s position iff \( K^0 \) is in John’s position.

**Proposition 4.10.** Let \( K \) be any 2-convex body with constant \( \alpha \) and volume 1. Then there exists a constant \( \lambda > 0 \) which depends on \( \alpha \) only, such that in L"owner’s minimal-volume outer ellipsoid position, \( \text{diam}(K) \leq c \frac{n^{1/2-\lambda}}{\lambda} \).

**Proof.** Apply Lemma 4.7 to \( K^0 \), which by duality is in John’s maximal-volume ellipsoid position. Then:

\[ M_2(K)\text{diam}(K) \leq T_2(X_K). \]

Since \( M_2(K) \geq \text{Vol.rad.}(K)^{-1} = 1 \) by Jensen’s inequality, it is enough to show that \( T_2(X_K) \) is bounded by \( c n^{1/2-\lambda} \). By Lemma 4.4, we know that there exists a \( p > 1 \) which depends on \( \alpha \) only, such that \( T_p(X_K) \leq 1/(p-1) \), so it remains to pass from type-\( p \) to type-2. But this is an easy consequence of a result by Tomczak-Jaegermann ([40]), who showed that it is enough to evaluate the type 2 constant of an \( n \)-dimensional Banach space on \( n \) vectors. If \( x_1, \ldots, x_n \) is any sequence in \( \mathbb{R}^n \), then by Hölder’s inequality:

\[ E\| \sum_{i=1}^n \varepsilon_i x_i \|_K \leq \left( \sum_{i=1}^n \| x_i \|_K^p \right)^{\frac{1}{p}} \leq \left( \frac{n^{p-1}}{p-1} \right)^{\frac{1}{p}} \left( \sum_{i=1}^n \| x_i \|_K^2 \right)^{\frac{1}{2}}. \]

Therefore \( T_2(X_K) \leq c \frac{n^{1/2-\lambda}}{\lambda} \), for \( \lambda = 1 - 1/p \). \( \square \)

We conclude this section by mentioning that the results of Section 2 imply that for 2-convex bodies, the isotropic position is a 1-regular M-position. Indeed, since the isotropic position is also a finite volume-ratio position, the second half of condition (4.4) is trivially satisfied. The first half is satisfied by the result from ([15] or [20, Proposition 5.4]), which shows that this is always the case for any isotropic body for which \( L_K \) is bounded. Note that
[11, Theorem 5.6] (which uses Dudley’s entropy bound) enables us to bound the mean-width of a convex body in an \(\alpha\)-regular M-position, which for a 1-regular position gives:

\[ M^*(K) \leq C\text{diam}(K)^{1/2} Vol.\text{rad.}(K)^{1/2}. \]

Since \(\text{diam}(K) \leq C\sqrt{n}L_K Vol.\text{rad.}(K)\) in isotropic position (e.g. \([29]\)), we conclude that \(M^*(K) \leq C(\alpha)n^{1/4}Vol.\text{rad.}(K)\) for any 2-convex body \(K\) with constant \(\alpha\) in isotropic position. It is still unclear to us whether the isotropic position is always a 2-regular M-position, which would imply (as above) that \(M^*(K) \leq C(\alpha)\log(n)Vol.\text{rad.}(K)\).

To summarize, we have seen the following implications for a 2-convex body:

- Minimal mean-width position implies essential isotropicity and a 2-regular M-position.
- John’s maximal-volume ellipsoid position implies finite volume-ratio position, essential minimal mean-width, 2-regular M-position and essential isotropicity.
- Löwner’s minimal-volume outer ellipsoid position implies “small-diameter”.
- Isotropic position implies finite volume-ratio position and 1-regular M-position.

5. Gaussian marginals

Similarly to the 2-convex case, we say that a convex body \(K\) is \(p\)-convex (with constant \(\alpha\)) if its modulus of convexity satisfies \(\delta_K(\epsilon) \geq \alpha\epsilon^p\) for all \(\epsilon \in (0,2)\). Let us also denote \(d_K = \text{diam}(K)\). It is well-known and easy to see (e.g. \([23]\) or follow the argument in Lemma 2.2) that the Gromov-Milman Theorem (2.2) immediately implies the following:

**Lemma 5.1.** Let \(K\) be a \(p\)-convex body with constant \(\alpha\) and of volume 1. For any 1-Lipschitz function \(f\) on \(K\) denote by \(\text{Med}(f)\) the median of \(f\), i.e. the value for which \(\text{Vol}\{x \in K; f(x) \geq \text{Med}(f)\} \geq 1/2\) and \(\text{Vol}\{x \in K; f(x) \leq \text{Med}(f)\} \geq 1/2\). Then:

\[ \text{Vol}\{x \in K; f(x) \geq \text{Med}(f) + t\} \leq 2\exp(-2\alpha n(t/d_K)^p). \]

Let us denote \(E(f) = \int_K f(x)dx\). As in [1], we deduce from Lemma 5.1 that \(|E(f) - \text{Med}(f)| \leq C d_K(\alpha n)^{-\frac{1}{p}}\). We therefore have:

\[ \text{Vol}\{x \in K; |f(x) - E(f)| \geq t + C d_K(\alpha n)^{-\frac{1}{p}}\} \leq 4\exp(-2\alpha n \left(\frac{t}{d_K}\right)^p), \]

and it is easy to check that this implies:

**Lemma 5.2.** With the same notations as in Lemma 5.1:

\[ \text{Vol}\{x \in K; |f(x) - E(f)| \geq t\} \leq 4\exp(-2\alpha n \left(\frac{t}{d_K}\right)^p). \]

Using this, it was shown in [1] that if \(K\) is an isotropic \(p\)-convex body (with constant \(\alpha\)) with \(|K| = 1\) and \(\text{diam}(K) \leq R\sqrt{n}\), then:

\[ \text{Vol}\left\{x \in K; \frac{|x|}{\sqrt{n}} - L_K \geq Rt\right\} \leq 4\exp(-2\alpha n t^p). \]

Choosing \(t = C(\frac{\log(n)}{\alpha n})^{1/p}\), this implies:

\[ \text{Vol}\left\{x \in K; \frac{|x|}{\sqrt{n}} - L_K \geq CR\left(\frac{\log(n)}{\alpha n}\right)^{1/p}\right\} \leq \frac{1}{n}. \]

(5.1)
The authors of [1] conclude that if $R \ll (\alpha n/\log(n))^{1/p}$, (5.1) implies a concentration of the volume of $K$ inside a spherical shell around a radius of $\sqrt{n}L_K$. It was shown in [1] that such a concentration implies that most marginals of the uniform distribution on $K$ will have an approximately Gaussian distribution (see Theorem 5.3 below). Unfortunately, our investigation of the case $p = 2$ shows that this condition on $R$ is not satisfied in general by isotropic 2-convex bodies, as demonstrated by Example 4.9. Nevertheless, Proposition 4.10 shows that in Löwner’s minimal-volume ellipsoid position, we do have $R \leq Cn^{1/2-\lambda}/\lambda$ where $\lambda$ depends only on the 2-convexity constant of $K$. In this case, the concentration result of [1] still holds, with the minor change that $c_1L_K \geq c_2$, e.g. [29]). Although $K$ is no longer isotropic, it is possible to generalize the argument in [1] to a body in arbitrary position. This is done in [28], where the following is shown:

**Theorem 5.3 (Generalization of [1]).** Let $K$ be a centrally-symmetric convex body in $\mathbb{R}^n$ of volume 1, and assume that for some $\rho > 0$ and $\epsilon < 1/2$:

$$
\text{Vol}\left\{ x \in K; \left| \frac{x}{\sqrt{n}} - \rho \right| \geq \epsilon \rho \right\} \leq \epsilon.
$$

(5.2) For $\theta \in S^{n-1}$ denote $g_\theta(s) = \text{Vol}\left( K \cap \left\{ s\theta + \theta^\perp \right\} \right)$ and let $\rho^2_\theta = \int_{-\infty}^{\infty} s^2 g_\theta(s) ds$. Denote the Gaussian density with variance $\rho^2$ by $\phi(s) = \frac{1}{\sqrt{2\pi}\rho} \exp\left( -\frac{s^2}{2\rho^2} \right)$ and let $H(\theta) = \sup_{t>0} \left| \int_t^1 g_\theta(s) ds - \int_t^1 \phi(s) ds \right|$. Then for any $0 < \delta < c$:

$$\sigma \left\{ \theta \in S^{n-1}; H(\theta) \leq \delta + 4\epsilon + \frac{c_1}{\sqrt{n}} \right\} \geq 1 - C_1 C_{\text{iso}}(K) \sqrt{n} \log n \exp\left( -\frac{c_2 n \delta^2}{C_{\text{iso}}(K)^2} \right),$$

(5.3)

where:

$$\rho_{\text{max}} = \max_{\theta \in S^{n-1}} \rho_\theta, \quad \rho_{\text{avg}} = \int_{S^{n-1}} \rho_\theta d\sigma(\theta), \quad C_{\text{iso}}(K) = \frac{\rho_{\text{max}}}{\rho_{\text{avg}}}.$$

**Remark 5.1.** As usual, it is easy to verify that $\rho_{\text{avg}}$ and $\rho$ above are equivalent to within absolute constants (since $\epsilon < 1/2$).

If $T$ is a volume preserving linear transformation such that $\tilde{K} = T(K)$ is isotropic, then clearly $\rho_{\text{max}} = \| T^{-1} \|_{\text{op}} L_K$, where $\| \cdot \|_{\text{op}}$ denotes the operator norm. Since $\rho_{\text{avg}}^2 \approx \frac{1}{n} \int_K |x|^2 dx \geq L_K^2$ (e.g. [29]), it follows that $C_{\text{iso}}(K) \leq C \| T^{-1} \|_{\text{op}}$. Hence, knowing that $r D_n \subset \tilde{K}$ and $K \subset RD_n$ would imply that $C_{\text{iso}}(K) \leq CR/r$. By Lemma 2.3 and Proposition 4.10, $c\sqrt{\alpha}\sqrt{n}L_K D_n \subset \tilde{K}$ and $K \subset Cn^{1-\lambda}/\lambda$ in Löwner’s position, where $\lambda > 0$ depends only on $\alpha$. We therefore have in this position:

$$C_{\text{iso}}(K) \leq \min\left( \frac{C n^{1/2-\lambda}}{\sqrt{\alpha} \lambda L_K}, C \sqrt{n} \right).$$

Hence, regardless of its a-priori diameter, by putting a 2-convex body $K$ with constant $\alpha$ in Löwner’s position, we deduce by Proposition 4.10, Lemma 5.2 and Theorem 5.3 that most marginals of $K$ are approximately Gaussian in the above sense, where the level of proximity ($\epsilon$ above) depends only on $\alpha$. Summarizing, we have:
Theorem 5.4. Let $K$ be a 2-convex body with constant $\alpha$ and volume 1. Assume that $K$ is in Löwner’s minimal-volume outer ellipsoid position. Then with the same notations as in Theorem 5.3 and with $\rho = \int_K |x| dx / \sqrt{n}$, we have for any $0 < \delta < c$:

$$\sigma \left\{ \theta \in S^{n-1}; H(\theta) \leq \delta + 4\epsilon + c \right\} \geq 1 - n^{5/2} \exp \left( -c_2 \alpha n^{2\lambda} \lambda^2 \delta^2 \right),$$

where $\epsilon = C \sqrt{\log n} \alpha^{-1/2} \lambda^{-1} n^{-\lambda}$ and $\lambda = \lambda(\alpha) > 0$ depends on $\alpha$ only.

Before concluding, we remark that placing a 2-convex body $K$ in Löwner’s position is just a convenient “pre-processing” step. In fact, in any position we always have at least one approximately Gaussian marginal (in the above sense); it just happens that in Löwner’s position we can show this for “most” marginals with respect to the Haar probability measure on the unit sphere, and this would equally be true in an arbitrary position by choosing a different measure (the one induced by the change of positions, for example). The reason is that the metric given by $H(\theta)$ in Theorem 5.3 is invariant under volume-preserving linear transformations. More precisely, given such a $T$, and any body $K$ and $\rho > 0$, it is immediate to check that:

$$\int_{-t}^t (g_{\rho T}(s) - \phi_\rho(s)) ds = \int_{-t}^t (\frac{g_{\rho}(s)}{\rho(\theta)} - \phi_\rho(\theta)(s)) ds,$$

so by Theorem 5.4 we can control the supremum over $t > 0$ of either expressions for at least one $\theta \in S^{n-1}$ if $K$ is a 2-convex body in Löwner’s position and $\rho = \int_K |x| dx / \sqrt{n}$.

Remark 5.2. After this work was written, it was proven by the second author Bo’az Klartag [21] that the “central limit property”, in the sense of Proposition 1.5, actually holds for all convex bodies. Note, however, that our quantitative estimates, for the case of 2-convex bodies, are essentially better.

References


E-mail address: bklartag@math.princeton.edu

**School of Mathematics, Institute of Advanced Study, Einstein Drive, Princeton, NJ 08540.**

E-mail address: emanuel.milman@weizmann.ac.il

**Department of Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel.**
CHAPTER 6

ON GAUSSIAN MARGINALS OF UNIFORMLY CONVEX BODIES

EMANUEL MILMAN

Submitted

Abstract. We show that many uniformly convex bodies have Gaussian marginals in most directions in a strong sense, which takes into account the tails of the distributions. These include uniformly convex bodies with power type 2, and power type $p > 2$ with some additional type condition. In particular, all unit-balls of subspaces of quotients of $L_p$ for $1 < p < \infty$ have Gaussian marginals in this strong sense. The same is true when $L_p$ is replaced by $S_p^m$, the $l_p$-Schatten class space. We also extend our results to arbitrary uniformly convex bodies with power type $p$, for $2 \leq p < 4$. These results are obtained by putting the bodies in (surprisingly) non-isotropic positions and by a new concentration of volume observation for uniformly convex bodies.

1. Introduction

In recent years, numerous results have been obtained of the following nature: let $X$ denote a uniformly distributed vector inside a centrally-symmetric convex body $K$ of volume 1 in $\mathbb{R}^n$. Let $X_\theta := \langle X, \theta \rangle$ denote its marginal in the direction of $\theta \in S^{n-1}$, where $S^{n-1}$ denotes the Euclidean unit sphere. Show that under suitable conditions on $K$, the distribution of $X_\theta$ is approximately Gaussian for most directions $\theta \in S^{n-1}$. Of course, the meaning of "approximately" and "most" need to be carefully defined, and vary among the different results.

To better illustrate this, consider the following examples. If $K = [-\frac{1}{2}, \frac{1}{2}]^n$, an $n$-dimensional cube, and $\theta = \frac{1}{\sqrt{n}}(1, \ldots, 1)$, the classical Central Limit Theorem asserts that $\langle X, \theta \rangle$ tends in distribution to a Gaussian with variance $\frac{1}{12}$. Of course this is false for all directions $\theta \in S^{n-1}$, as witnessed by the directions aligned with the cube’s axes, but does hold for most directions as measured by $\sigma$, the Haar probability measure on $S^{n-1}$. When $K$ is a volume 1 homothetic copy of the Euclidean ball $D_n$, the fact that (all) marginals are approximately Gaussian is classical, dating back to Maxwell, Poincaré and Borel (see [15] for a historical account). Other concrete bodies, such as the cross-polytope and simplex, were studied in [12]. Motivated by these and other results, it was conjectured by Antilla, Ball and Perissinaki [1] and Brehm and Voigt [12] (using different and in fact stronger formulations) that all convex bodies in $\mathbb{R}^n$ have at least one marginal which is approximately Gaussian, with the deviation tending to 0 as the dimension $n$ tends to $\infty$. In the broader context of general measures on $\mathbb{R}^n$ with finite second moment, Sudakov ([31])

Supported in part by BSF and ISF.
showed that most marginals are approximately the same mixture of Gaussian distributions. Under additional conditions on the covariance matrix of the measure in question, Diaconis and Freedman ([14]) showed that this mixture can be replaced by a proper Gaussian. A generalized version of both results was given by von Weizsäcker in [35].

In this note, we will focus on showing the existence of approximately Gaussian marginals for a rather wide class of symmetric convex bodies. Earlier results in this direction which have been most influential to our work include [1], [30] and [20]; other references are given later on. In those and previously mentioned results, approximately Gaussian marginals are found by requiring from $K$ that its volume be highly concentrated around a thin spherical shell of radius $\sqrt{n}\rho$, for some $\rho > 0$ and $\epsilon < 1/2$:

\begin{equation}
\mathrm{Prob}\left(\left|\frac{|X|}{\sqrt{n}} - \rho\right| \geq \epsilon \rho\right) \leq \epsilon.
\end{equation}

Usually, in order to obtain this type of volume concentration, the body $K$ is put in isotropic position, which is simply an affine image of $K$ of volume 1 for which $\text{Var}(X_\theta) = L_K^2$ for all $\theta \in S^{n-1}$ (it is well known that every full-dimensional body has an affine image which is isotropic). The constant $L_K$, called the isotropic constant of $K$, is affine invariant by definition. When $K$ is isotropic and satisfies (1.1), it is easy to see that $\rho$ must actually be close to $L_K$. When $\text{Var}(X_\theta) \leq C\rho^2$ for all $\theta \in S^{n-1}$, where $C > 0$ is some universal constant, we will say that $K$ is sub-isotropic. Following [20] but contrary to other approaches, and perhaps surprisingly, we will see in this note that it turns out to be more useful to put the body $K$ in some non-isotropic position (i.e. a volume-preserving affine image), for which we can show (1.1).

Let us denote the density function of $X_\theta$ by $g_\theta(s) := \text{Vol}\left(K \cap \{s\theta + \theta\}^\perp\right)$, and denote the average density over all possible directions by $g_{\text{avg}}(s) := \int_{S^{n-1}} g_\theta(s) d\sigma(\theta)$. Let $\rho_\theta^2$ denote the variance of the distribution corresponding to the density $g_\theta$, and set $\rho_{\text{max}} = \max_{\theta \in S^{n-1}} \rho_\theta$ and $\rho_{\text{avg}} = \int_{S^{n-1}} \rho_\theta d\sigma(\theta)$. Let $\phi_\rho(s) := \frac{1}{\sqrt{2\pi}\rho} \exp(-\frac{s^2}{2\rho^2})$ denote the Gaussian density with variance $\rho^2$. To emphasize that these notions depend on $K$, we will usually use $\rho_{\text{max}}(K)$ instead of $\rho_{\text{max}}$, etc. We reserve the symbols $C, C', C_1, C_2, c, c_1, c_2$ etc. to indicate positive universal constants, independent of all other parameters, whose value may change from one appearance to the next.

There are usually two steps in showing the existence of approximately Gaussian marginals: first, show that $g_{\text{avg}}$ is close to $\phi_\rho$, and then show that most densities $g_\theta$ are close to $g_{\text{avg}}$. Again, the meaning of “close to” and “most” vary between the results. In [1], the proximity between two even densities $f_1, f_2$ was interpreted in a rather weak sense, by using the Kolmogorov metric:

\begin{equation}
d_{Kol}(f_1, f_2) := \sup_{t \geq 0} \left| \int_{-t}^{t} f_1(s) ds - \int_{-t}^{t} f_2(s) ds \right|,
\end{equation}

which does not capture the similarity in the tail behaviour of the densities. We summarize the two steps from [1] into a single statement. In fact, our first remark in this note is that the argument of [1], originally derived for an isotropic body, applies to a body in arbitrary position, with some penalty accounting for the deviation from isotropic position, as measured by:

\[ C_{\text{iso}}(K) := \rho_{\text{max}}(K)/\rho_{\text{avg}}(K). \]
This more general statement, which was already used (without proof) in [20], reads as follows:

**Theorem 1.1** (Generalized from [1]). Assume that (1.1) holds for a centrally-symmetric convex body \( K \) in \( \mathbb{R}^n \). Then for any \( 0 < \delta < c \):

\[
\sigma \left\{ \theta \in S^{n-1}; d_{Kol}(g_{\theta}(K), \phi_\rho) \leq \delta + 4\epsilon + \frac{c_1}{\sqrt{n}} \right\} 
\geq 1 - C_1 C_{iso}(K) \sqrt{n} \log n \exp \left( -\frac{c_2 n \delta^2}{C_{iso}(K)^2} \right).
\]

(1.3)

Theorem 1.1 is proved in Section 2. We remark that it is easy to check that \( c_1 \rho_{avg} \leq \rho \leq c_2 \rho_{avg} \) (for some universal constants \( c_1, c_2 > 0 \)), whenever \( \rho \) satisfies (1.1), so we will sometimes use \( \rho_{max}(K) / \rho \) in place of the above definition of \( C_{iso}(K) \).

In [30], S. Sodin interpreted the proximity between two even densities \( f_1, f_2 \) in a much stronger sense, by measuring:

\[
d_{Sod}(f_1, f_2) := \sup_{0 \leq s \leq T} \left| \frac{f_1(s)}{f_2(s)} - 1 \right|,
\]

(1.4)

where \( T \) may be as large as some power of \( n \). Of course, this stronger notion requires a stronger condition on the concentration of volume inside \( K \):

\[
\text{Prob} \left( \left| \frac{|X|}{\sqrt{n}} - \rho \right| \geq t \rho \right) \leq A \exp(-Bn^{\delta\beta}),
\]

for all \( 0 \leq t \leq 1 \) and some \( A, B, \delta, \beta > 0 \). In that case, we summarize the two steps in [30] into the following single statement. The following formulation extends the result, originally formulated for bodies in sub-isotropic position, to convex bodies in arbitrary position.

**Theorem 1.2** ([30]). Let \( K \) denote a centrally-symmetric convex body in \( \mathbb{R}^n \) and assume that (1.5) holds. For \( 0 < \epsilon < c \) and \( \mu > 0 \) let:

\[
T = \rho \min \left( \left( \frac{cnC_{iso}(K)^{-2} \epsilon^4}{\log n + \log \frac{1}{\epsilon} + \mu} \right)^{1/6}, (c(A, B, \delta, \beta) \epsilon)^{\gamma/\delta} n^{\gamma} \right),
\]

where \( \gamma := \delta / (2 \max(\beta, 1)) \) and \( c(A, B, \delta, \beta) \) explicitly depend on \( A, B, \delta, \beta \). Then:

\[
\sigma \left\{ \theta \in S^{n-1}; d_{Sod}(g_{\theta}(K), \phi_\rho) \leq \epsilon \right\} \geq 1 - \exp(-\mu).
\]

(1.6)

We remark that it is possible to use other notions of proximity between densities besides (1.2) and (1.4) to measure the deviation from the Gaussian distribution. These include general weak law-convergence metrics (e.g. [35], [14]), the Kantorovic-Rubinstein metric (e.g. [31]), or the stronger \( l_1 \) and \( l_\infty \) norms (e.g. [15], [12],[11]). All of these notions of proximity, apart from the one given by (1.4), fail to take into account the tail behavior of the densities. Another exception, resembling (1.2) but taking into account the tails of the distributions, was used in [6] to show that for every convex body (and more generally log-concave measure) in isotropic position, the distribution of most marginals \( g_{\rho} \) is close to \( g_{avg} \), the average distribution of all marginals. Although this improves the second step from [1], where the same was shown using the weaker notion (1.2), it is harder to control the
expressions appearing in this result when deviating from isotropic position. We will therefore restrict our discussion to the notions given by (1.2) and (1.4), since these represent both ends of the proximity spectrum.

Concrete examples of classes of convex bodies for which Gaussian marginal results have been obtained can be roughly divided into two categories. The first category contains convex bodies possessing certain symmetries; these include the $l_p^n$ unit-balls ([1],[30]), more generally arbitrary unit-balls of generalized Orlicz norms ([36]), or [22]. The second category contains classes of uniformly convex bodies under certain restrictions ([1],[30],[20]). With any centrally-symmetric convex $K \subset \mathbb{R}^n$ we associate a norm $\|\cdot\|_K$ on $\mathbb{R}^n$. The modulus of convexity of $K$ is defined as the following function for $0 < \epsilon \leq 2$:

\[(1.7) \quad \delta_K(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|_K}{2} : \|x\|_K, \|y\|_K \leq 1, \|x - y\|_K \geq \epsilon \right\}. \]

Note that $\delta_K$ is affine invariant, so it does not depend on the position of $K$. A body $K$ is called uniformly convex if $\delta_K(\epsilon) > 0$ for every $\epsilon > 0$. A body $K$ is called “$p$-convex with constant $\alpha$” (see, e.g. [21, Chapter 1.e]), if for all $0 < \epsilon \leq 2$,

\[(1.8) \quad \delta_K(\epsilon) \geq \alpha \epsilon^p. \]

The restriction imposed on $p$-convex bodies is usually via an upper bound on the diameter of $K$ in isotropic position ([1]) or more generally in sub-isotropic position ([30]). For a 2-convex body $K$ with constant $\alpha$, this restriction on the diameter in isotropic position was recently removed by B. Klartag and the author in [20]. This was achieved by using Theorem 1.1, which as remarked above, holds in an arbitrary position. By putting $K$ in Löwner’s minimal diameter position, for which it was shown that $\text{diam}(K) \leq Cn^{1-\lambda}/\lambda$, where $\lambda > 0$ depends only on $\alpha$, the case of arbitrary 2-convex bodies was settled:

**Theorem 1.3** ([20]). Let $K \subset \mathbb{R}^n$ denote a 2-convex body with constant $\alpha$ and volume 1. Assume in addition that it is in Löwner’s minimal diameter position, and denote $\rho = \int_K |x|dx/\sqrt{n}$. Then (1.1) holds with:

\[ \epsilon = C\sqrt{\log n} \alpha^{-1/2} \lambda^{-1} n^{-\lambda}, \]

where $\lambda = \lambda(\alpha) > 0$ depends on $\alpha$ only. In addition, for any $0 < \delta < c$:

\[ \sigma \left\{ \theta \in S^{n-1} ; d_{K,d}(g_\theta(K), \phi_\rho) \leq \delta + 4\epsilon + \frac{c_1}{\sqrt{n}} \right\} \geq 1 - n^{\frac{\delta}{2}} \exp \left( -c_2 n^{2\lambda^2} \alpha^2 \lambda^2 \right). \]

Our second remark in this note is that the same argument works for arbitrary $p$-convex bodies ($p > 2$) which have a small type $s$ constant for large enough $s$ (see Section 3 for definitions). It is easy to show that such bodies have small diameter in Löwner’s position, and so the usual application of the Gromov-Milman concentration inside $p$-convex bodies (as in [1],[20]) gives the desired result. As for the case $p = 2$, the penalty $C_{iso}(K)$ needs to be handled in order to apply Theorem 1.1. We will denote by $T_s(X_K)$ the type-$s$ constant of the Banach space $X_K$ whose unit-ball is $K$.

**Theorem 1.4.** Let $K \subset \mathbb{R}^n$ denote a $p$-convex body with constant $\alpha$ and volume 1. Assume in addition that it is in Löwner’s minimal diameter position, and denote $\rho = \int_K |x|dx/\sqrt{n}$. 

Then for any $\frac{2p}{p+2} < s \leq 2$, (1.1) holds with:

$$\epsilon_s = CT_s(X_K)(\log n)^{\frac{1}{p} - \frac{1}{2} - \frac{1}{p} + \frac{1}{2}}.$$ 

In addition, for any $0 < \delta < c$:

$$\sigma\left\{ \theta \in S^{n-1}; d_{K\alpha}(g\theta(K), \phi_\rho) \leq \delta + 4\epsilon_s + \frac{c_1}{\sqrt{n}} \right\} \geq 1 - n^{\frac{5}{1}} \exp\left(-\frac{c_2n^2\delta^2 + 1}{\min(\rho, q)}\right).$$

When $K$ is the unit-ball of a subspace of quotient of $L_p$, for $1 < p < \infty$, a standard calculation (see Lemma 3.7) shows that the previous theorem may be applied to $K$. The same is true when $L_p$ is replaced by $S^m_p$, the Schatten class of $m$ by $m$ complex or real matrices, equipped with the norm $\|A\| = (\text{tr}(AA^*)^{p/2})^{1/p}$. This follows from the results of N. Tomczak-Jaegermann ([33]) about the equivalence between the modulus of convexity and type constants of $L_p$ and $S^m_p$. Plugging everything into the previous theorem gives:

**Corollary 1.5.** Let $K$ denote the unit-ball of an $n$-dimensional subspace of quotient of $L_p$ or $S^m_p$ for $1 < p < \infty$, and assume it has volume 1. Assume in addition that it is in Löwner’s minimal diameter position, and denote $\rho = \int_K |x|dx/\sqrt{n}$. Then (1.1) holds with:

$$\epsilon = C\sqrt{\rho}\left(\log n\right)^{\frac{1}{p}} n^{-\frac{1}{2}},$$

where $r = \max(p, q)$ and $q = p^*/p/(p-1)$. In addition, for any $0 < \delta < c$:

$$\sigma\left\{ \theta \in S^{n-1}; d_{K\alpha}(g\theta(K), \phi_\rho) \leq \delta + 4\epsilon + \frac{c_1}{\sqrt{n}} \right\} \geq 1 - n^{\frac{5}{1}} \exp\left(-\frac{c_2n^2\delta^2}{r} + 1\right).$$

With our extended formulation of Theorem 1.2 at hand, we can also give analogous results to those of Theorems 1.3, 1.4 and Corollary 1.5 using the stronger notion of proximity between densities (1.4). Indeed, for $p$-convex bodies as above, the Gromov-Milman argument already implies the stronger concentration assumption (1.5), and the penalty of $C_{iso}(K)$ appearing in Theorem 1.2 is handled exactly as for the former notion of proximity.

**Theorem 1.6.** Let $K \subset \mathbb{R}^n$ denote a $2$-convex body with constant $\alpha$ and volume 1. Assume in addition that $K$ is in Löwner’s minimal diameter position, and denote $\rho = \int_K |x|dx/\sqrt{n}$. Then (1.5) holds with $\delta = 2\lambda$, $\beta = 2$, $A = 4$ and $B = c\lambda^2$, where $\lambda = \lambda(\alpha) > 0$ depends on $\alpha$ only. In addition, (1.6) holds with:

$$T = \rho \min\left(\frac{\text{can}^{2\lambda^2} \epsilon^4}{\log n + \log \frac{1}{\epsilon} + \mu}, (c(\alpha)\epsilon)^{1/4}n^{\lambda/2}\right).$$

**Theorem 1.7.** Let $K \subset \mathbb{R}^n$ denote a $p$-convex body with constant $\alpha$ and volume 1. Assume in addition that $K$ is in Löwner’s minimal diameter position, and denote $\rho = \int_K |x|dx/\sqrt{n}$. Then (1.5) holds for any $\frac{2p}{p+2} < s \leq 2$ with $\delta = 1 - p/s + p/2$, $\beta = p$, $A = 4$ and $B = \alpha(c/T_s(X_K))^p$. In addition, (1.6) holds with:

$$T = \rho \min\left(\frac{\text{can}^{1+\frac{2\lambda}{2}} \epsilon^4}{\log n + \log \frac{1}{\epsilon} + \mu}, (c(p, \alpha, s)\epsilon)^{\frac{1}{2p}n^{\frac{1}{2} + \frac{1}{2p} - \frac{1}{p}}}\right).$$
Corollary 1.8. Let $K$ denote the unit ball of an $n$-dimensional subspace of quotient of $L_p$ or $S^m_p$ for $1 < p < \infty$, and assume it has volume 1. Assume in addition that $K$ is in L"owner’s minimal diameter position, and denote $\rho = \int_K |x|^2 dx / \sqrt{n}$. Then (1.5) holds for $\delta = \min(1, 2/q)\beta = \max(2, p), A = 4$ and $B = q^2(c p)^{p/2} - q^2$, where $q = p' = p/(p - 1)$. In addition, denoting $r = \max(p, q)$, (1.6) holds with:

$$T = \rho \min \left( \left( \frac{c n^{\frac{2}{p}} (r q)^{-1} \epsilon^4}{\log n + \log \frac{1}{\epsilon} + \mu} \right)^{1/6}, (c(p)\epsilon)^{\frac{1}{\max(p, q)} n^{\frac{1}{p}}} \right).$$

All of this is done in Section 3. In Section 4, we take on a different approach, which relies on the results of Bobkov and Ledoux from [7]. Contrary to other methods, which need to control the global Lipschitz constant of the Euclidean norm $|x|$ w.r.t. $\|\cdot\|_K$, the results in [7] enable us to average out the local Lipschitz constant of $|x|$ on $K$. Unfortunately, our estimate for this average enables us to deduce a result for $p$-convex bodies only in the range $2 \leq p < 4$.

Theorem 1.9. Let $K \subset \mathbb{R}^n$ denote a $p$-convex body with constant $\alpha$ for $2 \leq p < 4$, and assume it has volume 1. Assume in addition that it is in the position given by Theorem 4.6 below, and denote $\rho = \int_K |x|^2 dx / n$. Then (1.5) holds with $\beta = \frac{1}{2}, \delta = \frac{3}{8} - \frac{1}{2q}, A = 2$ and $B = cq^{\frac{1}{2}} / \min(f(p, \alpha), \log(1 + n))^{\frac{1}{2}}$, where $q = p' = p/(p - 1)$ and $f$ is some implicit function (given by Lemma 4.5). In addition, (1.6) holds with:

$$T = \rho \min \left( \left( \frac{c(\alpha n)^{\frac{1}{2}} \min(f(p, \alpha), \log(1 + n))^{-1} \epsilon^4}{\log n + \log \frac{1}{\epsilon} + \mu} \right)^{1/6}, (c(p, \alpha)\epsilon)^{\frac{1}{2} n^{\frac{1}{16}} - \frac{1}{4}} \right).$$

Note that for the range $2 \leq p < 4$, the latter Theorem holds without any assumptions on the diameter of the $p$-convex body (or the type constant of the corresponding space). Even for $p = 2$, this is an improvement over Theorem 1.3 which was proved in [20] and Theorem 1.6, since there an implicit function $\lambda = \lambda(\alpha)$ appears in several expressions and in particular in the exponent of $n$ (in Theorem 1.9 we can always replace $f$ by $\log (1 + n)$).

As a corollary, we strengthen Corollary 1.5 for unit-balls of subspaces of quotients of $L_p$ or $S^m_p$ with $1 < p < \frac{16}{13}$, since in this range, $r$ in Corollary 1.5 exceeds the value of $16/3$. These bodies are known to be 2-convex with constant $\alpha = c(p - 1)$ (see Lemma 3.7), so we may apply Theorem 1.9.

Corollary 1.10. Let $K$ be the unit-ball of an $n$-dimensional subspace of quotient of $L_p$ or $S^m_p$ for $1 < p \leq \frac{16}{13}$, and assume it has volume 1. Assume in addition that it is in the position given by Theorem 4.6 below, and denote $\rho^2 = \int_K |x|^2 dx / n$. Then (1.5) holds with $\beta = \frac{1}{2}, \delta = \frac{1}{8}, A = 2$ and $B = c(p - 1)^{\frac{1}{2}} / \log(1 + n)^{\frac{1}{2}}$. In addition, (1.6) holds with:

$$T = \rho \min \left( \left( \frac{c(\alpha n)^{\frac{1}{2}} \log(1 + n)^{-1} \epsilon^4}{\log n + \log \frac{1}{\epsilon} + \mu} \right)^{1/6}, (c(p)\epsilon)^{\frac{1}{2} n^{\frac{1}{16}}} \right).$$

Acknowledgments. I would like to sincerely thank my supervisor Prof. Gideon Schechtman for interesting discussions. I would like to thank Dr. Boaz Klartag for getting me
interested in the Central Limit Problem for convex bodies. I would also like to thank Profs. S. Bobkov and M. Ledoux for answering my questions.

2. Gaussian Marginals in Arbitrary Position

We dedicate this section to the proof of Theorem 1.1, which was already used in [20] to deduce Theorem 1.3, and which will be used in the next section for proving Theorem 1.4 and Corollary 1.5.

Proof of Theorem 1.1. We follow the proof in [1], emphasizing the necessary changes. Denote $G(t) = \int_{S^{n-1}} \int_{-t}^t g_\theta(s)dsd\sigma(\theta)$ and $\Phi_\rho(t) = \int_{-t}^t \hat{\phi}_\rho(s)ds$. It was shown in [1] that under the condition (1.2):

$$|G(t) - \Phi_\rho(t)| \leq 4\epsilon + \frac{c}{\sqrt{n}}$$

for any $t > 0$, and this is still valid for any position of $K$ since the isotropicity of $K$ was not used in the argument at all. Another important observation from [1], which holds regardless of position, is that for every $t > 0$, $\int_{-t}^t g_\theta(s)ds$ is a reciprocal of a norm. More precisely, denoting:

$$\|x\|_t = \frac{|x|}{\int_{-t}^t g_\frac{x}{|x|} (s)ds},$$

it was shown in [1] that $\|\cdot\|_t$ is a norm for any $t > 0$ and that:

$$a_t(\frac{x}{|x|})|x| \leq \|x\|_t \leq b_t(\frac{x}{|x|})|x|,$$

where $a_t, b_t$ satisfy for $\theta \in S^{n-1}$:

$$a_t(\theta) = c_1 \max(\frac{\rho_\theta}{t}, 1), \quad b_t(\theta) = c_2 \max(\frac{\rho_\theta}{t}, 1).$$

To conclude that given $t > 0$, the individual marginals $\int_{-t}^t g_\theta(s)ds$ are close to their spherical mean $G(t)$ (which is already known to be close to $\Phi_\rho(t)$), the authors of [1] invoke a classical result on concentration of Lipschitz functions around their mean: if $f : S^{n-1} \to \mathbb{R}$ is a $\lambda$-Lipschitz function then:

$$\sigma \left\{ \int_{S^{n-1}} f(\xi)d\sigma(\xi) \geq \delta \right\} \leq \exp(-Cn\delta^2/\lambda^2).$$

To this end, an estimate on the Lipschitz constant of $\int_{-t}^t g_\theta(s)ds$ is needed. Unfortunately, a straightforward application of the argument in [1] (as reproduced below) yields a Lipschitz constant of $C_{\rho_{\max} \rho_{\min}}$, where $\rho_{\min} = \min_{\theta \in S^{n-1}} \rho_\theta$, and this is not good enough for our purposes. We therefore modify the argument a little. For $0 < \gamma < 1$, let:

$$A_\gamma = \left\{ \theta \in S^{n-1}; \rho_\theta \geq (1 - \gamma)\rho_{\text{avg}} \right\}.$$

Since $\rho_\theta^2 = \int_K \langle x, \theta \rangle^2 dx$, it is clear that $\rho_\theta$ is a norm in $\theta$, and therefore its Lipschitz constant is bounded above by $\rho_{\max}$. Hence by (2.4):

$$\sigma(A_\gamma) \geq 1 - \exp \left( -\frac{Cn\gamma^2}{C_{\text{iso}}(K)^2} \right).$$
This means that for most directions, we can actually use \((1 - \gamma)\rho_{\text{avg}}\) as a lower bound on \(\rho_{\theta}\). Let \(a_t^j := c_1 \max((1 - \gamma)\rho_{\text{avg}}/t, 1)\), and define the modified norm \(\|x\|_t^j := \max(\|x\|_t, a_t^j|\theta|)\). Note that by (2.2) and (2.3), we did not alter the norm on \(\theta\) spread on the interval \([0, C]\), regardless of the value of \(|\theta|\) only outside the set \(-C\rho, C\rho\). Some absolute constant \(c > \delta > 0\) be given, and assume that \(\delta\) is not greater than some absolute constant \(c > 0\), so that we may define \(\gamma = C_0\delta < 1/2\). The fact that \(\rho_{\theta}\) is a norm implies (e.g. [25]) that \(\rho_{\text{max}} \leq C\sqrt{n}\rho_{\text{avg}}\), and therefore choosing \(C_0\) above big enough, we always have by (2.5), \(|G^\gamma(t) - G(t)| \leq \delta/2\). Hence:

\[
\sigma \left\{ \theta \in S^{n-1}; \left| \int_{-t}^t g_\theta(s) ds - G(t) \right| \geq \delta \text{ or } \theta \notin A_\gamma \right\} 
\leq \sigma \left\{ \theta \notin A_\gamma \right\} + \sigma \left\{ \theta \in S^{n-1}; \left| \int_{-t}^t g_\theta(s) ds - G(t) \right| \geq \delta - |G^\gamma(t) - G(t)| \right\} 
\leq \exp \left( - \frac{Cn\gamma^2}{C_{\text{iso}}(K)^2} \right) + 2 \exp \left( - \frac{Cn(\delta/2)^2(1 - \gamma)^2}{C_{\text{iso}}(K)^2} \right) \leq 3 \exp \left( - \frac{Cn\delta^2}{C_{\text{iso}}(K)^2} \right).
\]

Together with (2.1), and denoting \(H_\theta(t) = \left| \int_{-t}^t g_\theta(s) ds - \int_{-t}^t \phi_\rho(s) ds \right|\), we have for each \(t > 0\):

\[
\sigma \left\{ \theta \in S^{n-1}; H_\theta(t) \geq \delta + 4\epsilon + \frac{c}{\sqrt{n}} \text{ or } \theta \notin A_\gamma \right\} \leq 3 \exp \left( - \frac{Cn\delta^2}{C_{\text{iso}}(K)^2} \right).
\]

To pass from this estimate to one which holds for all \(t > 0\) simultaneously, we use the same argument as in [1], by “pinning” down \(H_\theta(t)\) at \(C\sqrt{n}\log(n)C_{\text{iso}}(K)\) points evenly spread on the interval \([0, C']\max(\rho, \rho_{\text{max}})\log(n)\). Since by our choice of \(\gamma\), for \(\theta \in A_\gamma\) we have \(\rho_{\theta} \geq \rho_{\text{avg}}/2\), it is easy to verify (as in [1]) that the Lipschitz constant of \(H_\theta(t)\) w.r.t. \(t\) is bounded above by \(C'/\rho_{\text{avg}}\) on \(A_\gamma\). By the remark after Theorem 1.1, we know that \(\rho\) and \(\rho_{\text{avg}}\) are equivalent to within universal constants, so the latter Lipschitz constant is bounded above by \(C'/\rho_{\text{avg}}\). Since the distance between two consecutive “pinned” points is \(C\rho_{\text{avg}}/\sqrt{n}\), this ensures that \(H_\theta(t)\) does not change by more than \(C''/\sqrt{n}\) between consecutive points, and this additional error is absorbed by the earlier error terms. There is no need to control \(H_\theta(t)\) for \(t \geq C'\max(\rho, \rho_{\text{max}})\log(n)\), since both \(\int_{-t}^\infty \phi_\rho(s) ds\) (Gaussian decay) and \(\int_{-t}^\infty g_\theta(s) ds\) (log-concavity of \(g_\theta\), see Lemma 4 in [1]), are smaller than \(C/\sqrt{n}\)
in that range, and this is again absorbed by the previous error terms. This concludes the proof. □

3. Concentration of Volume in Uniformly Convex Bodies with Good Type

In this section, we extend and strengthen the results from [20] to $p$-convex bodies with “good” type. Recall that the (Rademacher) type-$p$ constant of a Banach space $(X, \|\cdot\|)$ (for $1 \leq p \leq 2$), denoted $T_p(X)$, is the minimal $T > 0$ for which:

\[
\left( \mathbb{E} \left( \left\| \sum_{i=1}^{m} \varepsilon_i x_i \right\|^2 \right) \right)^{1/2} \leq T \left( \sum_{i=1}^{m} \|x_i\|^p \right)^{1/p}
\]

for any $m \geq 1$ and any $x_1, \ldots, x_m \in X$, where $\{\varepsilon_i\}$ are i.i.d. r.v.’s uniformly distributed on $\{-1, 1\}$ and $\mathbb{E}$ denotes expectation.

As explained in the Introduction, the existence of Gaussian marginals may be deduced using Theorems 1.1 or 1.2, once we show that the volume inside $K$ is concentrated around a thin spherical shell, in some controllable position of $K$. A fundamental observation on the concentration of volume inside uniformly convex bodies was given by Gromov and Milman in [17] (see also [2] for a simple proof). It states that if $K$ is uniformly convex with modulus of convexity $\delta_K$, and $T \subset K$ with $|T| \geq \frac{1}{2}|K|$, then for any $\varepsilon > 0$:

\[
\text{Vol} \left( (T + \varepsilon K) \cap K \right) \geq 1 - 2e^{-2n\delta_K(\varepsilon)}.
\]

It is easy to see that the latter is equivalent to the concentration around their mean of functions on $K$ which are Lipschitz w.r.t. $\|\cdot\|_K$.

Despite this attractive property of uniformly convex bodies, it is still a hard task to deduce concentration of volume around some spherical shell. The difficulty lies in the fact that for a convex body $K$, the function $|x|$ has a Lipschitz constant of $diam(K)$ w.r.t. $\|\cdot\|_K$, and this may be too big to be of use. In the next section, we describe an approach for which we will only need to control the average Lipschitz constant of $|x|$ on $K$, thereby eliminating the need to control $diam(K)$. In this section, as in [20], we use (3.1) in a direct manner, by putting $K$ in a position for which we have control over $diam(K)$. This will be ensured by the type condition on $K$.

We will use the following lemma, which is easy to deduce from (3.1) and the discussion above (see e.g. [1] or [20, Lemma 5.2]):

**Lemma 3.1.** Let $K \subset \mathbb{R}^n$ be a $p$-convex body with constant $\alpha$ and of volume 1. Then for any 1-Lipschitz (w.r.t. $\|\cdot\|_K$) function $f$ on $K$:

\[
\text{Vol} \left\{ x \in K; |f(x) - E(f)| \geq diam(K)t \right\} \leq 4 \exp(-2c_p \alpha nt^p).
\]

Denoting $\rho = \int_K |x|dx/\sqrt{n}$ and $R = diam(K)/\sqrt{n}$, we deduce:

\[
\text{Vol} \left\{ x \in K; \left| \frac{|x|}{\sqrt{n}} - \rho \right| \geq R t \right\} \leq 4 \exp(-2c_p \alpha nt^p).
\]

We see that in order to get some non-trivial concentration, we need to ensure that $R \ll n^{1/p}$. We will make use of the following lemma from [23] (which appeared first in an equivalent form in [13]):
Since in any position (e.g. \([24]\)):

\[ d_{\text{tm}} \text{ always satisfies:} \]

\[ K \text{ means that a 2-convex body } \]

\[ \text{depending solely on } \alpha < \lambda < \]

\[ \text{precisely, using the same notations as in } [20], \text{ it was shown that there exists a } 0 < K \]

\[ \text{for a 2-convex body } \]

\[ \text{we need to have a bounded type } \]

\[ \text{in addition that it is in Löwner’s minimal diameter position, and denote} \]

\[ \text{(3.5) } \text{Vol} \]

\[ \text{Plugging this into (3.3), we see that for such a body:} \]

\[ \text{Combining this with (3.2), we immediately have:} \]

\[ \text{Proposition 3.4. Let } K \subset \mathbb{R}^n \text{ be a } p\text{-convex body with constant } \alpha \text{ and of volume 1. Assume} \]

\[ \text{additionally that it is in Löwner’s minimal diameter position, and denote } \rho = \int_K |x| dx/\sqrt{n}. \text{ Then for any } 1 \leq s \leq 2 \text{ we have:} \]

\[ \text{(3.3) } \text{Vol} \left\{ x \in K; \frac{|x|}{\sqrt{n}} - \rho \geq t \right\} \leq 4 \exp \left( -2c_1 t^{\frac{p}{2}} \right). \]

\[ \text{In order to get a meaningful result, i.e. a positive power in the exponent of } n, \text{ we see that we need to have a bounded type } s \text{ constant } T_s(X_K) \text{ for } s > \frac{2p}{p+2}. \text{ It was shown in [20] that} \]

\[ \text{for a 2-convex body } K \text{ with constant } \alpha, \text{ this is always satisfied for some } s = s(\alpha) > 1. \text{ More precisely, using the same notations as in [20], it was shown that there exists a } 0 < \lambda < 1/2 \]

\[ \text{depending solely on } \alpha, \text{ such that for } s = \frac{1}{1-\lambda}, \text{ we have } T_s(X_K) \leq 1/\lambda. \text{ By Corollary 3.3, this means that a 2-convex body } K \text{ with constant } \alpha, \text{ having volume 1 and in Löwner’s position,} \]

\[ \text{always satisfies:} \]

\[ \text{(3.4) } d_{\text{tm}}(K) \leq Cn^{1-\lambda}/\lambda. \]

\[ \text{Plugging this into (3.3), we see that for such a body:} \]

\[ \text{(3.5) } \text{Vol} \left\{ x \in K; \frac{|x|}{\sqrt{n}} - \rho \geq t \right\} \leq 4 \exp \left( -ca \alpha^{2\lambda} \right). \]

\[ \text{Since in any position (e.g. [24]):} \]

\[ \text{(3.6) } \rho \geq c_1 L_K \geq c_2, \]

\[ \text{we get exactly the spherical concentration condition (1.5) needed for applying Theorem 1.2. It remains to evaluate } C_{\text{iso}}(K), \text{ appearing in Theorem 1.2. We argue as in [20], that} \]

\[ \rho_{\text{max}}(K) \text{ may be evaluated just by examining the radii of the circumscribing ball of } K \text{ and the inscribed Euclidean ball of } \tilde{K} = T(K), \text{ where } T \text{ is a volume preserving linear transformation such that } \tilde{K} \text{ is isotropic. Indeed, it is clear that } \rho_{\text{max}} = \|T^{-1}\|_{\text{op}} L_K, \text{ where} \]

\[ \|\cdot\|_{\text{op}} \text{ denotes the operator norm. And if } K \subset RD_n \text{ and } \tilde{K} \supset rD_n, \text{ where } D_n \text{ denotes the} \]

\[ \text{Euclidean unit ball, it is clear that } \|T^{-1}\|_{\text{op}} \leq R/r. \text{ In order to evaluate the radius of the} \]

\[ \text{inscribed ball of } \tilde{K}, \text{ we recall the following result from [20]:} \]
Lemma 3.5 ([20]). Let $K \subset \mathbb{R}^n$ denote a 2-convex body with constant $\alpha$ and volume 1. If $K$ is in isotropic position then:

\begin{equation}
(3.7) \quad c\sqrt{\alpha n} L_K D_n \subset K,
\end{equation}

implying in particular that $L_K \leq C/\sqrt{\alpha}$.

Using (3.4) and Lemma 3.5, we deduce that $\rho_{max}(K) \leq C n^{1/2-\lambda_1 \alpha^{-1/2}} \lambda^{-1}$. Using (3.6) and the remark after Theorem 1.1, we conclude that:

$$C_{iso}(K) \leq C n^{1/2-\lambda_1 \alpha^{-1/2}} \lambda^{-1}.$$  

Plugging everything into Theorem 1.2, Theorem 1.6 is deduced. We remark that Theorem 1.3 was deduced in [20] by choosing $t = c\sqrt{\log(n)} n^{-1/2}$ in (3.5) and applying Theorem 1.1.

For $p > 2$ the situation is different, because $2p/p+2 > 1$ and we cannot in general guarantee that given $p$ and $\alpha$, $T_s(X_K)$ is bounded even for $s = 2p/p+2$. We will therefore need to additionally impose some requirement on $T_s(X_K)$ for $s > 2p/p+2$. Once this is done, we deduce from (3.3), as for the case $p = 2$, the spherical concentration condition (1.5) needed for applying Theorem 1.2. In order to control the term $C_{iso}(K)$ in this case, we need to generalize Lemma 3.5 to the case of $p$-convex bodies. It is a mere exercise to repeat the proof in [20], which gives:

**Lemma 3.6.** Let $K \subset \mathbb{R}^n$ denote a $p$-convex body with constant $\alpha$ and volume 1. If $K$ is in isotropic position then:

\begin{equation}
(3.8) \quad c(\alpha n)^{1/p} L_K D_n \subset K,
\end{equation}

implying in particular that $L_K \leq C n^{1/2-\lambda_1 \alpha^{-1/2}} \lambda^{-1}$.

Arguing as above, this gives together with Corollary 3.3:

$$C_{iso}(K) \leq C n^{1/2-\lambda_1 \alpha^{-1/2}} \lambda^{-1} T_s(X_K).$$

Plugging this together with Proposition 3.4 into Theorem 1.2, Theorem 1.7 immediately follows. Choosing:

$$t = \frac{\log(n)^{1/p} T_s(X_K)}{ca^{1/p} n^{1/2-\lambda_1 /2}},$$

we deduce from (3.3) the spherical concentration condition (1.1) needed for applying Theorem 1.1, and so Theorem 1.4 is deduced.

It remains to deduce Corollaries 1.5 and 1.8 about unit-balls of subspaces of quotients of $L_p$ and $S_p^m$ for $1 < p < \infty$. With Theorems 1.4 and 1.7 at hand, we only need to evaluate these bodies’ $r$-convexity and type $s$ constants, for appropriately chosen $r$ and $s$. This is done in the following (essentially standard) lemma:

**Lemma 3.7.** Let $K \subset \mathbb{R}^n$ denote the unit-ball of a subspace of quotient of $L_p$ or $S_p^m$, for $1 < p < \infty$. Let $r = \max(p, 2)$, $s = \min(p, 2)$ and $q = p^*$. Then:

1. $K$ is $r$-convex with constant $\alpha(p) = C \min(p - 1, p^{-1} 2^{-p})$.
2. $T_s(X_K) \leq C \max(\sqrt{p}, \sqrt{q})$.
6. ON GAUSSIAN MARGINALS OF UNIFORMLY CONVEX BODIES

**Sketch of Proof.** We will sketch the proof of the \( L_p \) case. The proof of the \( S_p^\alpha \) case is exactly the same, since by the results of [33], these two classes have equivalent type, cotype and modulus of convexity (up to universal constants), and our proof of the \( L_p \) case will only depend on estimates for these parameters.

It is known (e.g. [21, Chapter 1.e]) that up to universal constants, \( L_p \) has the same modulus of convexity as \( l_p \), and that the latter space is \( r \)-convex with constant \( \alpha(p) \). By definition, this is passed on to any subspace of \( L_p \), and it is easy to see that the same holds for any quotient space (by passing to the dual and using the modulus of smoothness, see [20, Lemma 3.4]). Item (1) is thus shown.

To show item (2), first consider the case \( p \geq 2 \). Since \( L_q \) is 2-convex with constant \( q - 1 \), the dual \( L_p \) is 2-smooth (see [21, Chapter 1.e] or [20]) with constant \( \beta = c(q - 1)^{-1} \leq C_p \), and by the above discussion, the same is true for \( K \) as a unit-ball of a subspace of quotient of \( L_p \). It is standard (e.g. [20, Lemma 4.3]) that this implies that \( T_2(X_K) \leq C \sqrt{p} \leq C' \sqrt{q} \). When \( p < 2 \), we use a different argument. Denote by \( C_q(X) \) the cotype-\( q \) constant of a Banach space \( X \) and by \( \|\text{Rad}(X)\| \) the norm of the Rademacher projection on \( L_2(X_K) \) (see e.g. [25] for definitions). Assuming that \( K \) is the unit-ball of a subspace \( S \) of a quotient \( Q \) of \( L_p \), we have:

\[
T_p(X_K) = T_p(S) \leq T_p(Q) \leq C \|\text{Rad}(Q)\| C_q(Q^*) ,
\]

where the first inequality is immediate since type passes to subspaces, and the second one is known (e.g. [25]). But by duality, \( Q^* \) is a subspace of \( L_q \), and therefore inherits the cotype-\( q \) constant of \( L_q \), which is a universal constant (e.g. [25]). We conclude that \( T_p(X_K) \leq C \|\text{Rad}(Q)\| \). But again by duality \( \|\text{Rad}(Q)\| = \|\text{Rad}(Q^*)\| \leq \|\text{Rad}(L_q)\| \), since \( Q^* \) is a subspace of \( L_q \). We use the standard estimates \( \|\text{Rad}(L_q)\| \leq T_2(L_q) \leq C \sqrt{q} \) (e.g. [20]) to deduce that \( T_p(X_K) \leq C \sqrt{q} \). This concludes the proof.

Plugging this lemma into Theorems 1.4 and 1.7, Corollaries 1.5 and 1.8 are deduced.

4. CONCENTRATION OF VOLUME IN \( p \)-CONVEX BODIES FOR \( p < 4 \)

Let \( K \) denote a \( p \)-convex body in \( \mathbb{R}^n \). As already mentioned, it was first noticed by Gromov and Milman ([17]) that functions on \( K \) which are Lipschitz w.r.t. \( \|\cdot\|_K \) are in fact concentrated around their mean. This phenomenon has since been further developed by many authors (e.g. [28],[29],[2]). A common property to all of these approaches is that the level of concentration depends on the global Lipschitz constant of the function in question, even if in most places the function has a much smaller local Lipschitz constant. The starting point in the following discussion is the interesting results of Bobkov and Ledoux in [7], which overcome the above mentioned drawback.

Recall that the **entropy** of a non-negative function \( f \) w.r.t. a probability measure \( \mu \), is defined as:

\[
\text{Ent}_\mu(f) := \int f \log(f) d\mu - \int f d\mu \log(\int f d\mu).
\]

The expectancy and variance of \( f \) w.r.t. \( \mu \) are of-course:

\[
E_\mu(f) := \int f d\mu , \ Var_\mu(f) := E_\mu((f - E_\mu(f))^2).
\]
We will also use the following notation for \( q > 0 \):

\[
\Var^q_{\mu}(f) := E_{\mu}(|f - E_{\mu}(f)|^q).
\]

We will use \( \Ent_K(f) \), \( \Var_K(f) \) etc. when the underlying distribution \( \mu \) is the uniform distribution on \( K \). We also denote by \( \|\cdot\|^* \) the dual norm to \( \|\cdot\| \), defined as \( \|x\|^* = \sup \{|\langle x, y \rangle | : \|y\| \leq 1\} \). The following log-Sobolev type inequality was proved in [7, Proposition 5.4] (we correct here a small misprint which appeared in the original formulation):

**Theorem 4.1** ([7]). Let \( K \) be a \( p \)-convex body with constant \( \alpha \) and volume 1, and let \( q = p^* = p/(p - 1) \). Then for any smooth function \( f \) on \( K \):

\[
\Ent_K(|f|^q) \leq 2q \frac{2^n}{(\frac{n}{p} + 1)^q/n} \left( \frac{q}{\alpha} \right)^{q-1} \int_K (\|\nabla f\|^*_K)^q dx.
\]

When \( p = q = 2 \), it is classical that this log-Sobolev type inequality implies a Poincare-type inequality. Indeed, by applying Theorem 4.1 to \( f = 1 + \epsilon g \) and letting \( \epsilon \) tend to 0, we immediately have:

\[
\Var_K(g) \leq C \frac{\alpha n}{q} \int_K (\|\nabla g\|^*_K)^2 dx.
\]

More generally, it was shown in [8] that for any \( q \leq 2 \) and norm \( \|\cdot\| \), a \( q \)-log-Sobolev type inequality:

\[
\forall f \quad \Ent_{\mu}(|f|^q) \leq C \int \|\nabla f\|^q d\mu,
\]

always implies a \( q \)-Poincare-type inequality:

\[
\forall f \quad \Var^q_{\mu}(f) \leq C \frac{2^q}{\log^2} \int \|\nabla f\|^q d\mu.
\]

Although with this approach the additional term \( \frac{2^q}{\log^2} \) may not be optimal (as in the classical \( q = 2 \) case), universal constants do not play a role in our discussion. Applying this observation to the \( q \)-log-Sobolev inequality in Theorem 4.1 we deduce:

**Corollary 4.2.** With the same notations as in Theorem 4.1:

\[
\Var^q_K(f) \leq \frac{C}{(\alpha n)^{q-1}} \int_K (\|\nabla f\|^*_K)^q dx.
\]

Our goal will be to show some non-trivial concentration of the function \( g = |x|^2 \) around its mean, which is tantamount to the concentration of volume inside \( K \) around a thin spherical shell. As already mentioned, the advantage of the estimates in Theorem 4.1 and Corollary 4.2 is that they “average out” the local Lipschitz constant of \( f \) (w.r.t. \( \|\cdot\|_K \)) at \( x \in K \), which is precisely \( \|\nabla f(x)\|^*_K \). The usual way to deduce exponential concentration of \( g \) around its mean is via the Herbst argument, by applying Theorem 4.1 to the function \( f = \exp(\lambda g/q) \) (see [7] or [8]) and optimizing over \( \lambda \). Unfortunately, estimating the right-hand side of (4.1) for the function \( \exp(\lambda |x|^2/q) \) is a difficult task. An alternative way, which only produces polynomial concentration of \( g \) around its mean, is to apply Corollary 4.2 to the function \( f = g \) and use Markov’s inequality, in hope that estimating the right-hand side of (4.2) should be easier for \( g \) itself. We remark that it is possible to do the same with \( f = g \) in (4.1) and gain an additional logarithmic factor in the resulting concentration, but we avoid this for simplicity. We therefore start by applying Corollary 4.2 to the function \( f = |x|^2 \):
(4.3) \[ \text{Var}_{K}^q(|x|^2) \leq \frac{C'}{(\alpha n)^{q-1}} \int_K (\|x\|_K^*)^q dx. \]

In the following Proposition we estimate the right-hand side of (4.3). We denote by \( M^*(K) \) half the mean-width of \( K \), i.e. \( M^*(K) = \int_{S^{n-1}} \|\theta\|^*_K d\sigma(\theta) \). We also denote by \( SL(n) \) the group of volume preserving linear transformation in \( \mathbb{R}^n \).

**Proposition 4.3.** Let \( K \) be a \( p \)-convex body with constant \( \alpha \). Assume that \( K \) is isotropic and of volume 1, and set \( q = p^* = p/(p-1) \). Then for any \( T \in SL(n) \):

\[
(\text{Var}_{T(K)}^q(|x|^2))^{1/q} \leq \frac{C'}{(\alpha n)^{2/4}} n^{3/4} M^*(T^*T(K)) L_K.
\]

**Proof.** Since:

\[
\int_{T(K)} (\|x\|_{T(K)}^*)^q dx = \int_K (\|x\|_{T^{-1}T(K)}^*)^q dx,
\]

by (4.3) and a standard Lemma of Borell ([9]):

\[
(\text{Var}_{T(K)}^q(|x|^2))^{1/q} \leq \frac{C'}{(\alpha n)^{2}} \left( \int_{K} (\|x\|_{T^{-1}T(K)}^*)^q dx \right)^{1/q} \leq \frac{C''}{(\alpha n)^{2/2}} \int_{K} \|x\|_{T^{-1}T(K)}^* dx.
\]

Let us evaluate the integral on the right. First, notice that the contribution of \( \{ x \in K \setminus C \sqrt{n}L_K D_n \} \) to this integral is negligible. To show this, we turn for simplicity to a recent result of Grigoris Paouris ([26]), who showed that when \( K \) is in isotropic position:

\[
\text{Vol}(K \setminus C \sqrt{n}L_K t D_n) \leq \exp(-\sqrt{n}t)
\]

for all \( t \geq 1 \), hence:

\[
\int_{K \setminus C \sqrt{n}L_K D_n} \|x\|_{T^{-1}T(K)}^* dx \leq \exp(-\sqrt{n}) \text{diam}(T^*T(K)) \text{diam}(K).
\]

Since \( \text{diam}(T^*T(K)) \leq C_1 \sqrt{n}M^*(T^*T(K)) \) and \( \text{diam}(K) \leq C_2 nL_K \), we see that the latter integral is bounded by \( C n^{3/4} M^*(K) L_K \). We also denote by \( K' = T^*T(K) \), it remains to evaluate:

(4.4) \[ \int_{K \cap C \sqrt{n}L_K D_n} \|x\|_{K'}^* dx. \]

To this end, we apply a result of Bourgain ([10]) which uses the celebrated “Majorizing-Measures Theorem” of Fernique-Talagrand (see [32]), to deduce that the latter is bounded by \( C n^{3/4} M^*(K') L_K \). We remark that this is essentially the same argument which yields Bourgain’s well known bound on the isotropic constant \( L_K \leq C n^{1/4} \log(1+n) \). For completeness, we outline Bourgain’s argument. The idea is to write \( \|x\|_{K'}^* \) as \( \sup_{y \in K'} \langle y, x \rangle \), so (4.4) becomes an expectation on a supremum of a sub-Gaussian process. For \( y \in \mathbb{R}^n \), let \( H_y(x) = \langle y, x \rangle \) denote a r.v. on the probability space \( \Omega_H \), where \( x \) is uniformly distributed.
on $K \cap C\sqrt{n}L_K D_n$. For a real-valued r.v. $H$ on a probability space $(\Omega, d\omega)$ and $\alpha > 0$, let $\|H\|_{L^\alpha(\Omega)}$ be defined as:

$$\|H\|_{L^\alpha(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \exp((H(\omega)/\lambda)^\alpha) d\omega \leq 2 \right\}.$$  

A standard calculation shows that:

$$\|H_y\|_{L^2(\Omega_H)} \leq C_1 \sqrt{\|H_y\|_{L^4(\Omega_H)} \|H_y\|_{L^\infty(\Omega_H)}}$$

$$\leq C_2 \sqrt{\|H_y\|_{L^2(\Omega_H)} \|H_y\|_{L^\infty(\Omega_H)}} \leq C_3 n^{1/4} L_K |y|.$$  

Denoting $H_y = H_y/(C_3 n^{1/4} L_K)$, the latter implies that the process $\{H_y\}$ is sub-Gaussian w.r.t. the Euclidean-metric, and hence by the Majorizing-Measures Theorem:

$$E_{\Omega_H} \sup_{y \in K'} H_y' \leq C E_{\Omega_G} \sup_{y \in K'} G_y,$$

where $G_y(x) = \langle y, x \rangle$ is a r.v. on the probability space $\Omega_G$, where $x$ is a $n$-dimensional standard Gaussian. This implies that:

$$E_{\Omega_H} \sup_{y \in K'} H_y \leq C_4 n^{1/4} L_K n^{1/2} M^*(K'),$$

and a similar bound holds for (4.4), since the volume of $K \cap C\sqrt{n}L_K D_n$ is close to 1.  

It is easy to check (e.g. [24]) that $E_{T(K)}(|x|^2) = \int_{T(K)} |x|^2 dx \geq n L_K^2$, and therefore any time the bound in Proposition 4.3 is asymptotically smaller than $n L_K^2$, we can deduce a concentration result for $|x|^2$ on $K$. Unfortunately, we are unable to do so in the isotropic position, which is perhaps the most natural position for such concentration of volume to occur. For example, when $K$ is a 2-convex isotropic body (with constant $\alpha$), we cannot say much about $M^*(K)$: to the best of our knowledge, the best upper bound was given in [20], where it was shown that in isotropic position $M^*(K) \leq C(\alpha)n^{3/4}$, which is exactly the critical value we wish to be properly below. We mention here a question left open in [20], asking whether it is true that for 2-convex bodies, the isotropic position is in fact a 2-regular $M$-position; a positive answer would imply (see [20]) that $M^*(K) \leq C(\alpha)n^{1/2} \log(n)$, which would enable us to deduce concentration in isotropic position.

Proposition 4.3 was deliberately formulated in a way which enables us to work around this problem. We will use a $T \in SL(n)$ so that $M^*(T^*T(K))$ is minimal. In order to use Theorem 1.1, we will also need to control $C_{iso}(T(K))$, which amounts (as in the previous section) to controlling $\|T\|_{op}$. For 2-convex bodies, the relations between the isotropic, the John and the minimal mean-width positions, were studied in [20]. Recall that the John position of a convex body $K$ is defined as the (unique modulo orthogonal rotations) position with maximal radius of the inscribed Euclidean ball. We summarize the additional relevant results from [20] in the following:

**Lemma 4.4** ([20]). Let $K$ be a 2-convex body with constant $\alpha$ and volume 1.

1. If $K$ is in minimal mean-width position then:

$$M^*(K) \leq C \sqrt{n} \min \left( \frac{1}{\sqrt{\alpha}}, \log(1 + n) \right).$$


(2) In fact, the same estimate on $M^*(K)$ is valid in John’s position.

The latter easily generalizes to the case of general $p$-convex bodies. We sketch the argument for the following lemma (see [21] for definitions):

**Lemma 4.5.** Let $K$ be a $p$-convex body with constant $\alpha$ and volume 1. If $K$ is in minimal mean-width position then:

$$M^*(K) \leq \sqrt{n} \min(f(p, \alpha), C \log(1 + n)),$$

where $f$ is a function depending solely on $p$ and $\alpha$.

**Sketch of proof.** Recall that by the classical result of Figiel and Tomczak-Jaegermann on the $l$-position ([16]), we have that in the minimal mean-width position, $M^*(K) \leq C \sqrt{n} \|\text{Rad}(X_K)\|$ for a convex body $K$ of volume 1, where $\|\text{Rad}(X_K)\|$ denotes the norm of the Rademacher projection on $L_2(X_K)$ (see e.g. [25] for definitions). Since $K$ is $p$-convex with constant $\alpha$, it is classical ([21, Proposition 1.e.2]) that $K^*$ is $q$-smooth ($q = p^*$) with constant $\beta(\alpha, p)$, and therefore ([3, Theorem A.7]) has type $q$, with $T_q(X^*_K)$ depending only on $p$ and $\alpha$. Pisier showed in [27] that $\|\text{Rad}(X)\| = \|\text{Rad}(X^*)\|$ may be bounded from above by an (explicit) function of $T_q(X^*)$ when $q > 1$, which shows that $M^*(K) \leq \sqrt{n}f(p, \alpha)$. By another important result of Pisier (e.g. [25]), for an $n$-dimensional Banach space $X$ one always has $\|\text{Rad}(X)\| \leq C \log(1 + n)$, showing that $M^*(K) \leq \sqrt{n}C \log(1 + n)$. □

Combining Lemmas 3.6 and 4.5 with Proposition 4.3, we get a concentration result for $p$-convex bodies with $2 \leq p < 4$. The concentration will be for $T(K)$, the position which is ”half-way” (in the geometric mean sense) between the isotropic position $K$ and the minimal mean-width position $T^*T(K)$.

**Theorem 4.6.** Let $K$ be a $p$-convex body with constant $\alpha$ for $2 \leq p < 4$. Assume that $K$ is isotropic and of volume 1, and set $q = p^*$. Then there exists a position $T(K)$ with $T \in SL(n)$, such that:

1. $$\|T\|_{op} \leq C \frac{n^{\frac{1}{2p}}} {\alpha^{\frac{1}{2p}} L^\frac{1}{2}_K} \min(f(p, \alpha), \log(1 + n))^\frac{1}{2}.$$  

2. $$\langle \text{Var}^2_{T(K)}(|x|^2)^{1/4} \leq C n^{\frac{1}{2} + \frac{1}{2p}} \alpha^{-\frac{1}{2}} L_K \min(f(p, \alpha), \log(1 + n)) \rangle.$$  

3. Set $\rho^2 = \int_{T(K)} |x|^2 dx/n$. Then:

$$\text{Vol} \left\{ x \in T(K); \frac{|x|}{\sqrt{n}} - \rho \geq t \rho \right\} \leq 2 \exp \left( - \frac{cL_K^{1/2} \alpha^{1/2} n^{1/2 - \frac{1}{2p}} \alpha^{-1/2}} {\min(f(p, \alpha), \log(1 + n))^{1/2}} \right).$$

**Proof.** Since the isotropic and the minimal mean-width positions are defined up to orthogonal rotations, we may find a positive definite $T \in SL(n)$ so that $T^*T(K)$ is in minimal mean-width position, which by Lemma 4.5 and Proposition 4.3 gives (2). Since $\text{diam}(T^*T(K)) \leq C \sqrt{n}M^*(T^*T(K))$, we also have:

$$T^*T(K) \subset C n \min(f(p, \alpha), \log(1 + n))D_n.$$
By Lemma 3.6, this means that:

\[\|T^*T\|_{op} \leq C n^{\frac{1-p}{2}} \min(f(p, \alpha), \log(1+n)),\]

which gives (1). To deduce (3), we use the results of Bobkov [5] on the growth of \(L_p\) norms of polynomials. Note that the function \(g(x) = |x|^2 - n\rho^2\) is a polynomial of degree 2, so by [5, Theorem 1] there exists a universal constant \(C > 0\) such that:

(4.7) \[E_{T(K)} \left( \exp \left( \frac{|g|^{1/2}}{C E_{T(K)}(|g|^{1/2})} \right) \right) \leq 2.\]

Since \(E_{T(K)}(|g|^{1/2}) \leq E_{T(K)}(|g|^2)^{\frac{1}{2n}} = \Var_{T(K)}(|x|^2)^{\frac{1}{2n}}\), using the Chebychev-Markov inequality, (4.7) and (4.5), yields:

\[
\text{Vol} \left\{ x \in T(K); \frac{|x|}{\sqrt{n}} - \rho \geq t\rho \right\} \leq \text{Vol} \left\{ x \in T(K); |x|^2 - n\rho^2 \geq n\rho^2 t \right\} \\
= \text{Vol} \left\{ x \in T(K); |g(x)|^{1/2} \geq \sqrt{n}t\rho \right\} \leq 2 \exp \left( -\frac{\sqrt{n}t\rho}{C \Var_{T(K)}(|x|^2)^{1/2}} \right) \\
\leq 2 \exp \left( -\frac{\rho \sqrt{\frac{1}{2}} \rho \frac{1}{2}}{C' L_{K}^{1/2} \min(f(p, \alpha), \log(1+n))^{1/2}} \right).
\]

(3) immediately follows since always \(\rho \geq L_K\) (e.g. [24]). \qed

Remark 4.1.

(1) We see from (2) and (3) that we get a non-trivial concentration when \(q > \frac{4}{3}\), i.e. \(p < 4\). Of course this is due to the extra \(n^{\frac{1}{2}}\) term which we had in Proposition 4.3.

(2) For 2-convex bodies, we can slightly improve the estimate on \(\|T\|_{op}\) by taking \(T^* T(K)\) to be in John’s position. Indeed, by part (2) of Lemma 4.4, we will have the same estimate on \(M^*(T^* T(K))\) as the one used in the proof of Theorem 4.6. The advantage of using John’s position is that \(T^* T(K) \subset C_n D_n\), improving the estimate in (4.6), which was used to derive the bound on \(\|T\|_{op}\).

The advantage of this theorem over the previous concentration results for \(p\)-convex bodies in [1] or [20] is three-fold. In [1], the concentration was shown under certain assumptions on the diameter of the bodies, which is not satisfied for some bodies (as shown in [20] even for \(p = 2\)). In [20], this restriction on the diameter was removed for \(p = 2\), but the resulting concentration depended on an implicit function \(\lambda = \lambda(\alpha)\), which appeared in the exponent of \(n\). In Theorem 4.6 for the case \(2 \leq p < 4\), the restrictions on the diameter of the bodies are removed, the dependence of the concentration on \(\alpha\) is explicit, and this dependence is not in the exponent in any of the expressions.

Since it is well know that \(L_K \geq c\) (e.g. [24]), Theorem 4.6 yields a concentration of the form (1.5) required to apply Theorem 1.2 to \(T(K)\). It remains to evaluate \(C_{iso}(T(K))\), taking into account the remark after Theorem 1.1. Since \(\rho_{avg} \geq c\rho \geq cL_K\), using (1) from
Theorem 4.6 and $L_K \geq c$, we have:

$$C_{iso}(T(K)) = \frac{\rho_{\max}}{\rho_{\avg}} \leq \frac{\|T\|_{op} L_K}{cL_K} \leq C n^{\frac{1}{2}} \alpha^{-\frac{1}{2}} \min(f(p, \alpha), \log(1 + n))^{\frac{1}{2}}.$$ 

Plugging everything into Theorem 1.2, we deduce Theorem 1.9. Corollary 1.10 is deduced by using the estimates given in Lemma 3.7.

**Remark 4.2.** Sasha Sodin has brought to our attention that a recent result of S. Bobkov ([4]) shows that all our concentration results for uniformly convex bodies in fact imply isoperimetric inequalities for these bodies (with respect to the Euclidean norm).

**Remark 4.3.** At the time this work was completed, Bo'az Klartag announced in [18] the following solution to the Central Limit Problem for convex bodies: for every isotropic convex body in $\mathbb{R}^n$:

$$\sigma(\theta \in S^{n-1}; d_{TV}(g_\theta(K), \phi_{L_K}) \leq \varepsilon_n) \geq 1 - \delta_n,$$

where $d_{TV}(f, g) = \int_{-\infty}^{\infty}|f(s) - g(s)| \, ds$ is the total-variation metric between the measures given by the densities $f, g$, and $\varepsilon_n, \delta_n$ are two series decreasing to 0. This result is in fact applicable to all log-concave probability measures with full dimensional support in $\mathbb{R}^n$. In addition, for suitable $k$ increasing with $n$, the existence of $k$-dimensional marginals which are approximately Gaussian was also shown. We remark that the metric $d_{TV}$ is much weaker than Sodin’s strong notion of proximity $d_{Sod}$, and shares the drawback of many other metrics in the sense that it does not capture the similarity of the tails of the densities, as they become very small. For log-concave densities, it is known (see [11]) that $d_{TV}$ and $d_{Kol}$ are equivalent in the sense that $d_{TV} \leq h_1(d_{Kol}) \leq h_2(d_{TV})$ for some known functions $h_1, h_2$, so our results for $d_{Kol}$ may be translated to the total-variation metric. We also remark that Klartag’s estimates yield a very slow logarithmic rate of convergence to the Gaussian law $\varepsilon_n \leq C \sqrt{\frac{\log \log n}{\log n}}$, which should be compared with our polynomial rates. Consequently, Klartag’s result does not give any information on the similarity between $g_\theta$ and $\phi_{L_K}$ outside the interval $I = [-CL_K \log \log n, CL_K \log \log n]$, since $\int_{\mathbb{R} \setminus I} h \leq 1/ \log n$ for $h = g_\theta, \phi_{L_K}$. As noted by Klartag, it is unknown whether this rate of convergence is sharp, but this seems unlikely; in fact, several authors (e.g., [1]) conjecture that the rate should be polynomial in $n$. In a very recent progress in this direction, Klartag [19] has indeed improved his logarithmic estimates to polynomial ones.

**References**


E-mail address: emanuel.milman@weizmann.ac.il

Department of Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel.
CHAPTER 7

ISOPERIMETRIC INEQUALITIES FOR UNIFORMLY LOG-CONCAVE MEASURES AND UNIFORMLY CONVEX BODIES

EMANUEL MILMAN\(^\dagger\) AND SASHA SODIN\(^\dagger\)

Submitted

Abstract. We prove an isoperimetric inequality for uniformly log-concave measures and for the uniform measure on a uniformly convex body. These inequalities imply the log-Sobolev inequalities proved by Bobkov and Ledoux [12] and Bobkov and Zegarlinski [13]. We also recover a concentration inequality for uniformly convex bodies, similar to that proved by Gromov and Milman [22].

1. Introduction

Let \( V = (\mathbb{R}^n, \|\cdot\|) \) be a normed space, and let \( \mu \) be a probability measure on \( V \) with density \( f = \exp(-g) \), \( g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \). If \( g \) is convex, the function \( f \) and the measure \( \mu \) are called log-concave. Log-concave functions and measures boast many important properties (cf. Borell [14], Bobkov [9] et cet.)

In this note, we study more restricted classes of measures. Let

\[ \delta : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}, \]

and consider the following condition:

\[ (1.1) \quad \frac{g(x) + g(y)}{2} - g\left(\frac{x + y}{2}\right) \geq \delta(\|x - y\|). \]

Example 1.1. The log-concavity condition corresponds to \( \delta \equiv 0 \).

By analogy with uniformly convex bodies (cf. Subsection 1.2.2), we define the modulus of convexity \( \delta_{g,\|\cdot\|} \) of \( g \) with respect to the norm \( \|\cdot\| \) as:

\[ \delta_{g,\|\cdot\|}(t) := \inf \left\{ \frac{g(x) + g(y)}{2} - g\left(\frac{x + y}{2}\right) : \|x - y\| \geq t \text{ and } g(x), g(y) < \infty \right\}. \]

If \( \delta_{g,\|\cdot\|}(t) > 0 \) for all \( t > 0 \), we say that \( f \) and \( \mu \) are uniformly log-concave, and that \( g \) is uniformly convex. Obviously, this notion does not depend on the choice of the norm \( \|\cdot\| \).

\(^\dagger\) Research was supported in part by the European Network PHD, MCRN–511953.
It is easy to check that $\delta_{g,\|\cdot\|}(t)/t$ is always a non-decreasing function of $t$; therefore in the sequel we consider measures $\mu$ satisfying (1.1) with respect to a function $\delta$ such that

\[
\delta(t) > 0, \quad t > 0
\]

\[
t \mapsto \delta(t)/t \quad \text{is non-decreasing.}
\]

**Example 1.2.** Let $\|\cdot\| = |\cdot|$ be the Euclidean norm, and let $\delta(t) = t^2/8$. Then (1.1) holds iff $\mu$ has log-concave density with respect to the standard Gaussian measure; in other words, if $\mu$ satisfies the Bakry–Émery curvature-dimension condition $\text{CD}(1, +\infty)$ (cf. Bakry and Émery [2]; recall that the usual log-concavity of $\mu$ is equivalent to $\text{CD}(0, +\infty)$).

**Remark 1.3.** The condition (1.1) is translation invariant. Therefore one may extend it to measures on an affine space $\mathbb{A}^n$ on which $V$ acts by translations; note that both sides of (1.1) are still defined. This point of view will be convenient in Section 2.

1.0. **Assumptions and Notations.** Unless mentioned otherwise, the sets in this note are Borel subsets of $\mathbb{R}^n$, and the measures are Borel measures on $\mathbb{R}^n$.

The Lipschitz norm of a map $T : V_1 \to V_2$ between two normed spaces $V_i = (X_i, \|\cdot\|_i)$, $i = 1, 2$, is defined as:

\[
\|T\|_{\text{Lip}} = \sup_{x, y \in X_1, x \neq y} \frac{\|T(x) - T(y)\|_2}{\|x - y\|_1}.
\]

$T$ is called Lipschitz if $\|T\|_{\text{Lip}} < \infty$. If

\[
\sup_{x, y \in K, x \neq y} \frac{\|T(x) - T(y)\|_2}{\|x - y\|_1} < +\infty
\]

for any compact subset $K \subset X_1$, $T$ is called locally Lipschitz.

A Borel map $T : V_1 \to V_2$ is said to push a measure $\mu$ on $V_1$ forward to a measure $\lambda$ on $V_2$ (notation: $T_*\mu = \lambda$) if $\mu(T^{-1}(B)) = \lambda(B)$ for every $B \subset X_2$.

If $\mu$ is a probability measure on $V = (X, \|\cdot\|)$, the Minkowski boundary measure associated with $\mu$ (and $\|\cdot\|$) is defined by:

\[
\mu^+_{\|\cdot\|}(A) = \liminf_{\varepsilon \to 0} \frac{\mu(A_{\varepsilon, \|\cdot\|}) - \mu(A)}{\varepsilon}, \quad A \subset X,
\]

where

\[
A_{\varepsilon, \|\cdot\|} = \{x \in X \mid \exists y \in A, \|x - y\| < \varepsilon\}
\]

is the $\varepsilon$-extension of $A$ in the metric induced by $\|\cdot\|$. In addition, we denote:

\[
\widehat{\mu}(A) = \min(\mu(A), 1 - \mu(A))
\]

for all $A \subset X$. Lastly, we denote the Lebesgue measure on $\mathbb{R}^n$ by $\text{mes}_n$.

1.1. **Isoperimetric inequalities.** The first topic of this note is an isoperimetric inequality for $\mu$. In the setting of Example 1.2 (and actually in a much more abstract one), Bakry and Ledoux proved [3] the following isoperimetric inequality:
Theorem (Bakry – Ledoux). If the measure $\mu$ satisfies (1.1) with $\| \cdot \| = | \cdot |$ and $\delta(t) = t^2/8$, then for any $A \subset \mathbb{R}^n$:

(1.5) \[ \mu^+(A) \geq \phi \left( \Phi^{-1} \left( \tilde{\mu}(A) \right) \right) . \]

Here as usual $\phi(t) = \frac{1}{\sqrt{2\pi}} \exp(-t^2/2)$ and $\Phi(t) = \int_{-\infty}^{t} \phi(s) \, ds$.

This theorem is a generalisation of the isoperimetric inequality for the Gaussian measure, proved by Sudakov, Tsirelson, and Borell [32, 15]. In [10], Bobkov gave a proof of the Bakry–Ledoux inequality using the localisation technique; the latter was introduced by Gromov and Milman [22] and developed by Kannan, Lovász and Simonovits [28], [24] (see also Gromov [21, §3.27]). We extend Bobkov’s approach to the general case (1.1) and prove:

Theorem 1.1. Suppose $\mu$ satisfies (1.1) and (1.2). Then

(1.6) \[ \mu^+(A) \geq C_\delta \tilde{\mu}(A) \gamma \left( \log \frac{1}{\mu(A)} \right) \text{ for all } A \subset \mathbb{R}^n , \]

where:

\[
C_\delta = \frac{e - 1}{2e \max(2\delta(\int_0^{+\infty} \exp(-2\delta(t))dt), 1)} , \quad \gamma(t) = \frac{t}{\delta^{-1}(t/2)} , \quad \mu(A) = \min(\mu(A), 1 - \mu(A)) .
\]

Corollary 1.2. Let $\delta(t) = \alpha t^p$ for $p \geq 2$ and $\alpha > 0$ in the setting of the previous theorem. Then:

(1.7) \[ \mu^+(A) \geq c \alpha^{1/p} \mu(A) \log^{1 - 1/p} \left( \frac{1}{\mu(A)} \right) , \]

where $c > 0$ is a universal constant (independent of $p$).

Remark 1.4. Note that $p$ can not be less than 2; this follows from a second-order Taylor expansion of $g$ in (1.1).

Remark 1.5. For $p = 2$, Corollary 1.2 recovers the Bakry–Ledoux Theorem up to a universal constant: indeed,

\[ \phi(\Phi^{-1}(t)) \leq C' t \sqrt{\log 1/t} , \quad 0 \leq t \leq 1/2 . \]

1.2. Application: Uniformly convex bodies. As before, let $V = (\mathbb{R}^n, \| \cdot \|)$ be a normed space. The volume measure $\lambda = \lambda_V$ on the unit ball of $V$ is defined by:

(1.8) \[ \lambda = \frac{\text{mes}_n(\{ \| x \| \leq 1 \})}{\text{mes}_n(\{ \| x \| \leq 1/2 \})} ; \]

it arises naturally in geometric applications.

We would like to prove an isoperimetric inequality for $\lambda$, with respect to the norm $\| \cdot \|$. It is easy to see that $\lambda$ never satisfies the condition (1.1) with $\delta > 0$. Therefore we follow the approach introduced by Bobkov and Ledoux [12] and define an auxiliary measure $\mu$ that satisfies (1.1).
1.2.1. **p-uniformly convex bodies.** Choose $p \geq 2$, and let $\mu$ be the measure with density:

$$
\exp(\|x\|^p) \over \Gamma(1 + n/p) \text{mes}_n(\{\|x\| \leq 1\})
$$

with respect to the Lebesgue measure.

**Proposition** (Bobkov – Ledoux). There exists a map $S : V \to V$ such that $S^* \mu = \lambda$ and $$
\|S\|_{L_p} \leq C (\Gamma(1 + n/p))^{-1/\alpha},
$$
where $C > 0$ is a universal constant.

It is clear that Lipschitz maps preserve isoperimetric inequalities, so we may first establish one for $\mu$. The condition (1.1) for $\mu$, with $\delta(t) = \alpha t^p$, reads as

$$
\|x\|^p + \|y\|^p - \|x + y\|^p \geq \alpha \|x - y\|^p
$$

for all $x, y \in \mathbb{R}^n$.

This is one of the definitions of a $p$-uniformly convex norm (cf. Pisier [31]).

**Example 1.6.** The $\ell_q$ norm $\| \cdot \|_q, 1 < q < \infty$, satisfies (1.10) with

$$
p = \begin{cases} 2, & q < 2 \\ q, & q \geq 2 \end{cases}, \quad \alpha = \begin{cases} q^{-1} \frac{4}{q}, & q < 2 \\ 2^{-q}, & q \geq 2 \end{cases}.
$$

In fact, the same estimates holds for the space $L_q$. The case $q \geq 2$ is due to Clarkson [16] (see also Hanner [23]), while the case $q < 2$ follows from an unpublished argument of Ball and Pisier (see Ball, Carlen and Lieb [6]).

Therefore, if $\| \cdot \|$ is $p$-uniformly convex with coefficient $\alpha$ (that is, if (1.10) holds), we can apply Corollary 1.2 and deduce (1.7). Combining with the Bobkov – Ledoux proposition above, we obtain the following:

**Theorem 1.3.** Suppose the space $V$ is $p$-uniformly convex with constant $\alpha$ (that is, satisfies (1.10)); let $\lambda$ be the uniform measure on the unit ball of $\| \cdot \|$ (as in (1.8)). Then for any $A \subset \mathbb{R}^n$:

$$
\lambda^+_{\| \cdot \|}(A) \geq C \alpha^{1/p} n^{1/p} \lambda(A) \log^{1-1/p} \frac{1}{\lambda(A)},
$$

where $C > 0$ is a universal constant.

This theorem continues the study of isoperimetric properties of $p$-uniformly convex bodies by Bobkov and Zegarlinski [13, Ch. 14]. In particular, when $\lambda(A)$ is not exponentially small in the dimension, the inequality in Theorem 1.3 improves the bound in [13, Theorem 14.6].

1.2.2. **General uniformly convex bodies.** We also generalise the above results to arbitrary uniformly convex spaces. Recall that the modulus of convexity $\delta_V : [0, 2] \to [0, 1]$ of a normed space $V = (X, \| \cdot \|)$ is defined as:

$$
\delta_V(\varepsilon) = \inf \left\{ 1 - \| x + y \| : \| x \|, \| y \| \leq 1, \| x - y \| \geq \varepsilon \right\}.
$$

The space is called uniformly convex if $\delta_V(\varepsilon) > 0$ for all $\varepsilon > 0$. From the works of Figiel [18], Figiel-Pisier [19] and Pisier [31], it is known that if:

$$
\delta_V(\varepsilon) \geq \alpha \varepsilon^p \text{ for all } \varepsilon \in [0, 2],
$$

then
7. ISOPERIMETRIC INEQUALITIES

then (1.10) holds with \( \alpha = \min(c, \alpha'/2^p) \), and that if (1.10) holds then (1.12) holds with \( \alpha' = \alpha/p \) (here \( c > 0 \) is a universal constant). A space is therefore \( p \)-uniformly convex if either (1.10) or (1.12) hold, it is however important to specify which definition one uses if the dependence on \( p \) is of interest.

In Section 4 we derive the following proposition from the results of Figiel-Pisier [19]:

**Proposition 1.4.** For all \( x, y \in X \) such that \( \|x\|^2 + \|y\|^2 \leq 2 \), one has:

\[
\frac{\|x\|^2 + \|y\|^2}{2} - \left\| \frac{x+y}{2} \right\|^2 \geq c \delta_V \left( \frac{\|x-y\|}{4} \right),
\]

where \( c > 0 \) is a universal constant.

Returning to the case \( X = \mathbb{R}^n \), choose \( \mu \) to be the probability measure with density:

\[
f(x) = \frac{1}{Z} \exp\left( -\frac{n}{c} \|4x\|^2 \right) 1 \left\{ \|x\| \leq \frac{1}{4} \right\}
\]

with respect to the Lebesgue measure, where \( Z > 0 \) is a scaling factor. Proposition 1.4 clearly implies that \( \mu \) is uniformly log-concave, so we can apply Theorem 1.1 and deduce an isoperimetric inequality for \( \mu \). To transfer this inequality to the measure \( \lambda_V \), we need to extend the Bobkov–Ledoux proposition of the previous subsection. Our next observation, which may be of independent interest, does precisely that.

**Definition.** A map \( T : \mathbb{R}^n \to \mathbb{R}^n \) is called radial if it maps every ray to itself in a monotone way; that is, if for every \( x \neq 0 \)

\[
\begin{align*}
T(\mathbb{R}_+ x) &\subset \mathbb{R}_+ x \\
T|_{\mathbb{R}_+ x} : \mathbb{R}_+ x &\to \mathbb{R}_+ x \text{ preserves the order on } \mathbb{R}_+ x.
\end{align*}
\]

Let \( d\mu = f d\text{mes}_n \) be an even log-concave probability measure (with log-concave density \( f \)). Denote

\[
K_f = \left\{ x \in \mathbb{R}^n; \ n \int_0^{+\infty} f(rx) r^{n-1} dr \geq 1 \right\} ;
\]

It is not hard to see (cf. Proposition 3.1) that there exists a canonical radial map \( T_f \) pushing forward \( \mu \) to the restriction \( \lambda \) of the Lebesgue measure to \( K_f \).

K. Ball showed [5] that \( K_f \) is a symmetric convex body; in other words, the unit ball of a norm \( \| \cdot \|_{K_f} \). In Section 3 we prove the following result (in a slightly more general form):

**Theorem 1.5.** Let \( \mu = f d\text{mes}_n \) be an even log-concave probability measure (with log-concave density \( f \)); let \( \lambda \) denote the restriction of the Lebesgue measure on \( K_f \), and let \( T = T_f \) denote the canonical radial map such that \( T_* \mu = \lambda \). Then as a map \( T : V \to V \) where \( V = (\mathbb{R}^n, \| \cdot \|_{K_f}) \), we have \( \|T\|_{\text{Lip}} \leq C f(0)^{1/n} \), where \( C > 0 \) is a universal constant.

**Remark 1.7.** The Bobkov-Ledoux proposition above is a particular case of the last Theorem (up to another universal constant). We provide the details at the end of Subsection 3.2.

In Section 4 we apply Theorems 1.1 and 1.5 to deduce the following:
Theorem 1.6. Let $V = (\mathbb{R}^n, \|\cdot\|)$ be a uniformly convex space, and let $\delta = \delta_V$ denote its modulus of convexity. Let $\lambda = \lambda_V$ denote the uniform measure on the unit-ball of $V$ (as in (1.8)) and let $A \subset \mathbb{R}^n$. If $\tilde{\lambda}(A) \geq \exp(-2\delta(1/4)n)$, then:

$$\lambda(A) \geq c' C_{n,\delta} \frac{\tilde{\lambda}(A) \log \frac{1}{\lambda(A)}}{\delta^{-1} \left( \frac{1}{2n} \log \frac{1}{\lambda(A)} \right)},$$

where:

$$\Delta_{n,\delta} = \frac{e - 1}{2e \max(n\delta(\int_{1/4}^{1/4} \exp(-2n\delta(t))dt), 1)},$$

and $c' > 0$ is a universal constant.

Note that when $\delta(t) = \alpha t^p$ ($p \geq 2$), Theorem 1.6 recovers Theorem 1.3 up to a universal constant, for sets whose measure is not exponentially small (in $n$).

1.3. Connection to functional inequalities and concentration. In this subsection we study some corollaries of the isoperimetric inequalities of the form (1.6) and (1.7).

1.3.1. Concentration. It is well-known that an isoperimetric inequality can be equivalently rewritten in global form. It will be convenient to use this in the following formulation (see Bobkov and Zegarlinski [13, p. 46] for an equivalent form):

Proposition 1.7. Let $\mu$ be a probability measure on $\mathbb{R}^n$ satisfying

$$\mu(A) \geq \tilde{\mu}(A) \gamma \left( \log \frac{1}{\mu(A)} \right)$$

for every Borel set $A \subset \mathbb{R}^n$ and some continuous function $\gamma : [\log 2, +\infty) \to \mathbb{R}_+$. Then for any Borel set $B \subset \mathbb{R}^n$ and any $\varepsilon > 0$:

$$1 - \mu(B_{\varepsilon,\|\cdot\|}) \leq \exp \left( -h_{1/\mu(B)}(\varepsilon) \right),$$

where:

$$h_a(x) = \int_{\log 1/a}^{x} \frac{dy}{\gamma(y)};$$

for $y < \log 2$, $\gamma(y)$ should be interpreted as $\gamma(\log \frac{1}{1-\exp(-y)})$.

Conversely, if $\mu$ satisfies (1.17) for any Borel set $B \subset \mathbb{R}^n$, then (1.16) holds.

Corollary 1.8. Let $\mu$ be a measure on $\mathbb{R}^n$ such that for all $A \subset \mathbb{R}^n$:

$$\mu(A) \geq c_0 \tilde{\mu}(A) \log^{1-1/p} \frac{1}{\mu(A)}.$$

Then for every $B \subset \mathbb{R}^n$, $\mu(B) \geq 1/2$, and every $\varepsilon > 0$,

$$1 - \mu(B_{\varepsilon,\|\cdot\|}) \leq \exp \left\{ - \left[ \log^{1/p} \frac{1}{1-\mu(B)} + c_0 \varepsilon \right]^p \right\}. $$
In Subsection 5.1 we combine Proposition 1.7 and Corollary 1.8 with the results of the previous subsections, to deduce a concentration inequality for uniformly convex bodies. Then we compare this inequality with the Gromov–Milman theorem [22].

For completeness, we prove Proposition 1.7 in Subsection 5.2.

1.3.2. Functional inequalities. An isoperimetric inequality can be written in a functional form; this was brought forth by Maz’ya, Federer, and Fleming [29, 17] in the early 60’s and later adapted by Bobkov and Houdré [11] to the context of probability measures.

**Proposition (Bobkov–Houdré).** Let \( \mu \) be a probability measure on a normed space \((\mathbb{R}^n, \| \cdot \|)\), and let \( I : [0, 1/2] \to \mathbb{R}_+ \) be an increasing continuous function such that \( I(0) = 0 \). The following are equivalent:

1. For any Borel set \( A \subset \mathbb{R}^n \),
   \[
   \mu_{\| \cdot \|}(A) \geq I(\mu(A)) ;
   \]
2. For any locally Lipschitz function \( F : \mathbb{R}^n \to [0, 1] \) such that
   \[
   \mu\{F = 1\} \geq t \in (0, 1/2) \quad \text{and} \quad \mu\{F = 0\} \geq 1/2 ,
   \]
   we have:
   \[
   \int \|\nabla F\|_* d\mu \geq I(t) ,
   \]
   where
   \[
   \|\nabla F\|_* = \lim_{y \to x} \sup |F(y) - F(x)| / \|y - x\| .
   \]

Let us focus on the case \( I(t) = c_0 t \log^{1/q} 1/t \), where \( 1/q = 1 - 1/p \). We have the following:

**Proposition 1.9.** Suppose a probability measure \( \mu \) on \((\mathbb{R}^n, \| \cdot \|)\) satisfies

\[
\mu_{\| \cdot \|}(A) \geq c_0 \mu(A) \log^{1/q} \frac{1}{\mu(A)}
\]

for all \( A \subset \mathbb{R}^n \). Then:

1. For any locally Lipschitz function \( F : \mathbb{R}^n \to [0, 1] \) satisfying (1.22), we have:
   \[
   \int \|\nabla F\|_* d\mu \geq c_0 t \log^{1/q} 1/t ;
   \]
2. for any locally Lipschitz function \( F : \mathbb{R}^n \to [0, 1] \) satisfying (1.22), we have:
   \[
   \int \|\nabla F\|_*^q d\mu \geq c c_0^q t \log 1/t ,
   \]
   where \( c > 0 \) is a universal constant;
3. for any locally Lipschitz function \( F : \mathbb{R}^n \to \mathbb{R}_+ \),
   \[
   \int \|\nabla F\|_*^q d\mu \geq c' c_0^q \int F^q \log \frac{F^q}{\int F^q d\mu} d\mu ,
   \]
   where \( c' > 0 \) is a universal constant.
Of course, 1. follows from the previous proposition (and in fact, (1.25) is equivalent to (1.16)). Then, 1. implies 2. via standard arguments that we reproduce for completeness in Section 5. Finally, 2. is equivalent to 3. (up to universal constants); this is a reformulation of the arguments developed by Bobkov and Zegarlinski [13, Chapter 5.] in the language of capacities put forth by Barthe and Roberto [7].

The inequality (1.27), called a $q$-log-Sobolev inequality, was studied by Bobkov and Ledoux [12] and Bobkov and Zegarlinski [13]. In particular, part 3. of the last proposition extends Theorem 16.3 in [13]. Combining it with Theorems 1.1 and 1.3, we recover the $q$-log-Sobolev inequalities proved by Bobkov and Ledoux in [12], up to universal constants.

Acknowledgments. The authors Emanuel Milman and Sasha Sodin thank their supervisors Gideon Schechtman and Vitali Milman for their guidance and support. Part of this work was done while the authors enjoyed the hospitality of the Henri Poincaré Institute in Paris.

2. An Isoperimetric Inequality

2.1. Reduction to one dimension. This subsection is based on an argument that was introduced by Gromov and Milman [22] to reduce the spherical isoperimetric inequality to a certain one-dimensional fact; see also Gromov [21, §3.2.27]. The corresponding argument in the affine case was developed by Kannan, Lovász and Simonovits [28, 24], who also coined the term ‘localisation lemma’.

We formulate the localisation lemma in terms of $\mu$-needles, as put forth by S. Bobkov; this corresponds to convex descendants in [21]. It will be natural to work in an $n$-dimensional affine space $A^n$ (cf. Remark 1.3).

Let $V = (\mathbb{R}^n, \| \cdot \|)$ be a normed space acting by translations on an affine space $A^n$. Let $\mu$ be a probability measure on $A^n$ such that $\mu(H) = 0$ for every affine hyperplane $H \subset A^n$.

Definition. A (probability) measure $\sigma$ supported on an affine line $L \subset A^n$ (and not on any point) is called a $\mu$-needle if

$$\sigma = \lim_{k \to +\infty} \mu\left|\frac{C_k}{\mu(C_k)}\right|$$

is the weak limit of the scaled restrictions of $\mu$ to convex sets

$$C_1 \supset C_2 \supset C_3 \cdots, \quad \mu(C_k) > 0.$$  

Localisation principle: global form. Let $\mu$ be a probability measure on $A^n$ such that $\mu(H) = 0$ for every affine hyperplane $H \subset \mathbb{R}^n$; let $a, b \in (0, 1)$, $\varepsilon > 0$. If every $\mu$-needle $\sigma$ supported on an affine line $L_\sigma$ satisfies:

$$\sigma(A'_\varepsilon) \geq b \text{ for every } A' \subset L_\sigma \text{ such that } \sigma(A') = a,$$

then also:

$$\mu(A_\varepsilon) \geq b \text{ for every } A \subset A^n \text{ such that } \mu(A) = a.$$

This is essentially the first step in [28]. It will be more convenient to obtain an infinitesimal form of this localisation principle. Given an isoperimetric inequality in the general form:

$$\mu^+(A) \geq I(\mu(A)),$$
where $I : [0, 1/2] \to \mathbb{R}_+$ is a continuous function, we may of course write $I(a) = a\gamma(\log 1/a)$ for some continuous function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$, obtaining the form in (1.16). By Proposition 1.7, a local isoperimetric inequality of the form (1.16) is equivalent to the global inequality (1.17). Applying this twice, we deduce the following:

**Localisation principle: local form.** Let $\mu$ be a probability measure on $\mathbb{A}^n$ such that $\mu(H) = 0$ for every affine hyperplane $H \subset \mathbb{A}^n$, and let $I : [0, 1/2] \to \mathbb{R}_+$ denote a continuous function. If every $\mu$-needle $\sigma$ supported on an affine line $L_\sigma$ satisfies:

$$\sigma^\perp_\parallel(A') \geq I \left( \bar{\sigma}(A') \right) \text{ for every } A' \subset L_\sigma,$$

then also:

$$\mu^\perp_\parallel(A) \geq I \left( \bar{\mu}(A) \right) \text{ for every } A \subset \mathbb{A}^n.$$

To complete the reduction to one dimension, let us show that “if $\mu$ is uniformly log-concave, its needles are also uniformly log-concave”. The following lemma extends [10], [21, §3.2.27, Ex. (e)].

**Lemma 2.1.** Let $V = (\mathbb{R}^n, \|\|)$ be a normed space acting by translations on an affine space $\mathbb{A}^n$, and let $\delta : \mathbb{R}_+ \to \mathbb{R}_+$. If a measure $\mu$ on $\mathbb{A}^n$ satisfies the uniform log-concavity condition (1.1) with respect to $\delta$ and $\|\|$, then every $\mu$-needle $\sigma$ supported on an affine line $L \subset \mathbb{A}^n$ satisfies (1.1) with respect to $\delta$ and the restriction of $\|\|$ to the tangent space $L - L$.

**Sketch of proof.** Let $f$ denote the density of $\mu$ with respect to the Lebesgue measure on $\mathbb{A}^n$. $\mu$ satisfies (1.1), hence $f$ is in particular log-concave. Log-concave functions have many regularity properties (cf. Borell [14]). It was shown in [28, Lemma 2.5] that when $f$ is lower semi-continuous, any $\mu$-needle supported on an affine line $L$ has density $f|_L \phi$ with respect to the Lebesgue measure on $L$, for some log-concave function $\phi$. It is easy to check that the proof remains valid if $f$ is a log-concave function, or alternatively, replace $f$ by an equivalent lower semi-continuous density $f'$ which satisfies (1.1).

Now, $f|_L$ satisfies (1.1) with respect to $\delta$ and $\|\|_L - L$. Since $\phi$ satisfies (1.1) with respect to $\delta'$, it follows that $f|_L \phi$ satisfies (1.1) with respect to $\delta + \delta' = \delta$ and $\|\|_L - L$. □

By the lemma, it is sufficient to prove Theorem 1.1 for $n = 1$. In this case, we only need the following property of one-dimensional uniformly log-concave measures:

**Lemma 2.2.** Let $V = (\mathbb{R}^n, \|\|)$, and assume that $g : V \to \mathbb{R} \cup \{+\infty\}$ satisfies (1.1). Assume in addition that $a$ is a minimum point of $g$. Then:

$$g(x) - g(a) \geq 2\delta(x - a),$$

for all $x \in \mathbb{R}^n$.

**Proof.** If $g(x) = +\infty$, the claim is trivial. Otherwise, apply (1.1) with $g = a$. Then:

$$\delta(x - a) \leq \frac{g(x) - g(a)}{2} + g(a) - g \left( \frac{x + a}{2} \right) \leq \frac{g(x) - g(a)}{2},$$

where we used the fact that $a$ is a minimum point of $g$ in the last inequality. □
We will prove the isoperimetric inequality for one-dimensional measures $\mu$ with density $f = \exp(-g)$, where $g$ satisfies (2.5). Any norm on $\mathbb{R}^1$ is Euclidean, hence without loss of generality $\| \cdot \| = | \cdot |$. Therefore Theorem 1.1 is reduced to the following proposition (note the factor 2 that we drop between (2.5) and (2.6) to simplify the notation).

**Proposition 2.3.** Let $\sigma$ denote a probability measure on $\mathbb{R}$ with density $f$. Assume that $f = \exp(-g)$, where $g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is a convex function with minimum at 0 and such that:

\begin{equation}
(2.6) \quad g(x) - g(0) \geq \delta(|x|)
\end{equation}

for all $x \in \mathbb{R}$, and $\delta : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ satisfies (1.2). Then:

\begin{equation}
(2.7) \quad \sigma^+(A) \geq C_{\delta} \sigma(A) \gamma \left( \frac{1}{\sigma(A)} \right)
\end{equation}

for any $A \subset \mathbb{R}$, where

$$C_{\delta} = \frac{e - 1}{2e \max(\delta(\int_0^{+\infty} \exp(-\delta(t))dt), 1)}; \quad \gamma(t) = \frac{t}{\delta^{-1}(t)}.$$

### 2.2. Proof of the one-dimensional inequality

**Lemma 2.4.** The function $\gamma$ is non-decreasing. The function $x \gamma \left( \log \frac{1}{x} \right)$ is strictly increasing on $[0, 1/e]$.

**Proof.** The first part follows since $\delta(x)/x$ is non-decreasing by our assumption (1.2). For the second part, write:

$$x \gamma \left( \log \frac{1}{x} \right) = \frac{x \log \frac{1}{x}}{\delta^{-1}(\log \frac{1}{x})},$$

so the claim follows since $\delta$ (and hence $\delta^{-1}$) is non-decreasing, whereas $x \log \frac{1}{x}$ is increasing on $[0, 1/e]$. \hfill $\Box$

Now denote:

$$M_{\delta} = \int_0^{+\infty} \exp(-\delta(x)).$$

**Lemma 2.5.**

$$\exp(-g(0)) \geq (2M_{\delta})^{-1}.$$

**Proof.** Using $\int f(x)dx = 1$ and (2.6):

$$1 = \int_{\mathbb{R}} \exp(-g(x))dx \leq \exp(-g(0)) \int_{\mathbb{R}} \exp(-\delta(|x|))dx. \hfill \Box$$

**Lemma 2.6.** If $g$ is a convex function on $\mathbb{R}$ with minimum at 0, then for all $x > 0$:

$$\int_x^{+\infty} \exp(-g(y))dy \leq \frac{x}{g(x) - g(0)} \exp(-g(x)).$$
**Proof.** By convexity, it follows that for all \( y \geq x \):
\[
g(y) \geq \frac{g(x) - g(0)}{x} (y - x) + g(x).
\]
Using this to bound \( \int_x^\infty \exp(-g(y)) \, dy \) from above, the claim follows. \( \square \)

Given a finite measure \( \mu \) on \( \mathbb{R} \), we denote by \( m(\mu) \) its median, i.e. the unique value \( m \) for which \( \mu((-\infty, m]) \geq \mu(\mathbb{R})/2 \) and \( \mu([m, \infty)) \geq \mu(\mathbb{R})/2 \).

**Lemma 2.7.** For any finite log-concave measure \( d\mu = f \, dx \) on \( \mathbb{R} \),
\[
f(m(\mu)) \geq \frac{1}{2} \max_{x \in \mathbb{R}} f(x).
\]

**Proof.** Without loss of generality, assume \( m = m(\mu) > 0 \), \( f(0) = \max f \) and \( f(m) < f(0) \). Then \( f \) is non-increasing on \( \mathbb{R}_+ \). Replace \( \mu \) with \( \mu|_{\mathbb{R}_+} \); then the left-hand side of (2.8) may only decrease, whereas the right-hand side retains its value.

Now replace \( f \) by a log-affine function \( f_1 \) on \( \mathbb{R}_+ \) such that \( f_1(0) = f(0) \) and \( f_1(m) = f(m) \). In other words \( f_1(x) = \exp(-ax + b)|_{\mathbb{R}_+} \), and our assumptions imply that \( a > 0 \). Setting \( d\mu_1 = f_1 \, dx \), \( \mu_1 \) is a finite measure. Then \( f_1 \leq f \) on \([0, m]\) and \( f_1 \geq f \) on \([m, +\infty)\); hence \( m(\mu_1) \geq m(\mu) \) and \( f(m(\mu)) = f_1(m(\mu_1)) \geq f_1(m(\mu_1)) \).

Finally,
\[
f_1(m(\mu_1)) = \frac{1}{2} \max_{x \in \mathbb{R}_+} f_1(x);
\]
this concludes the proof. \( \square \)

**Proof of Proposition 2.3.** By a general result of Bobkov ([8, Proposition 2.1]) on extremal isoperimetric sets of log-concave densities, it is enough to verify (2.7) on sets \( A \) of the form \((-\infty, a]\) and \([b, \infty)\). Given a point \( x \in \mathbb{R} \), denote \( A = [x, \infty) \) if \( x \geq 0 \) and \( A = (-\infty, x] \) if \( x < 0 \). We will show that the set \( A \) satisfies:
\[
\sigma^+(A) \geq C_g \sigma(A) \gamma \left( \log \frac{1}{\sigma(A)} \right),
\]
and this will conclude the proof. Assume w.l.o.g. that \( x \geq 0 \), since our hypotheses are symmetric about the origin.

First, recall that by another result of Bobkov ([9, Proposition 4.1]), a log-concave probability measure \( \mu \) with density \( f \) on \( \mathbb{R} \) always satisfies the following Cheeger-type isoperimetric inequality:
\[
\mu^+(A) \geq 2f(m) \min(\mu(A), 1 - \mu(A)),
\]
where \( m \) is the median of \( \mu \). Together with Lemma 2.7, this implies:
\[
(2.9) \quad \sigma^+(A) \geq \exp(-g(0)) \sigma(A).
\]
Loosely speaking, this Cheeger-type inequality will take care of the case when \( \sigma(A) \) is large. The case when \( \sigma(A) \) is small will be handled by Lemma 2.6, which, together with the assumption (2.6) and the fact that \( \delta \) is increasing, imply that for any \( x > 0 \):
\[
\sigma(A) = \int_x^\infty \exp(-g(y)) \, dy \leq \frac{\delta^{-1}(g(x) - g(0))}{g(x) - g(0)} \exp(-g(x)).
\]
Recalling the definition of $\gamma$ and denoting $\sigma^+_{\text{max}} = \exp(-g(0))$, this means:

\[
(2.10) \quad \sigma(A) \leq \frac{\sigma^+(A)}{\gamma(g(x) - g(0))} = \frac{\sigma^+(A)}{\gamma(\log \frac{\sigma^+_{\text{max}}}{\sigma^+(A)})}.\]

This inequality is almost what we need, and the rest of the proof will be dedicated to replacing $\sigma^+$ with $\sigma$ inside the $\gamma$ function.

More formally, we distinguish between five cases.

1. $\tilde{\sigma}(\bar{A}) \geq c_\delta$, where $c_\delta \leq 1/e$ depends solely on $\delta$ and will be determined later. In this case, by (2.9) and Lemma 2.5:

   \[
   \sigma^+(A) \geq \exp(-g(0))\tilde{\sigma}(\bar{A}) \geq \frac{1}{2M_\delta} \tilde{\sigma}(A).\]

   The function $\gamma$ is non-decreasing by Lemma 2.4, therefore

   \[
   \sigma^+(A) \geq \frac{1}{2M_\delta \gamma(\log \frac{1}{c_\delta})} \tilde{\sigma}(A) \gamma(\log \frac{1}{\sigma(A)}).
   \]

2. $1 - \sigma(A) = \tilde{\sigma}(\bar{A}) < c_\delta$ and $g(x) - g(0) < \log \frac{1}{c_\delta}$. Using (2.6):

   \[
   \sigma(A) \leq \int_0^\infty \exp(-g(y))dy \leq \exp(-g(0)) \int_0^\infty \exp(-\delta(y))dy,
   \]

   and since $g(x) - g(0) < \log \frac{1}{c_\delta}$ we conclude that:

   \[
   1 - c_\delta < \sigma(A) \leq \frac{1}{c_\delta} \exp(-g(x))M_\delta = \frac{M_\delta}{c_\delta} \sigma^+(A).
   \]

   By Lemma 2.4, $x\gamma(\log \frac{1}{x})$ is monotone increasing on $[0, 1/e]$. Since $\tilde{\sigma}(A) < c_\delta \leq 1/e$, we conclude that:

   \[
   \sigma^+(A) \geq \frac{(1 - c_\delta)c_\delta \gamma(\log \frac{1}{c_\delta})}{M_\delta \gamma(\log \frac{1}{c_\delta})} \geq \frac{(1 - c_\delta)}{M_\delta \gamma(\log \frac{1}{c_\delta})} \tilde{\sigma}(A) \gamma(\log \frac{1}{\sigma(A)}).
   \]

3. $\sigma(A) = \tilde{\sigma}(\bar{A}) < c_\delta$ and $g(x) - g(0) < \log \frac{1}{c_\delta}$. As in 2.:

   \[
   1 - \sigma(A) = \int_{-\infty}^0 \exp(-g(y))dy + \int_0^x \exp(-g(y))dy \leq \exp(-g(0))M_\delta + \exp(-g(0))x \leq \frac{1}{c_\delta} \exp(-g(x))(M_\delta + x).
   \]

   Using (2.6) and the inequality $g(x) - g(0) < \log \frac{1}{c_\delta}$,

   \[
   x \leq \delta^{-1}(g(x) - g(0)) \leq \delta^{-1}(\log \frac{1}{c_\delta}).
   \]

   Hence:

   \[
   1 - c_\delta \leq 1 - \sigma(A) \leq \frac{M_\delta + \delta^{-1}(\log \frac{1}{c_\delta})}{c_\delta} \sigma^+(A).
   \]

   Now choose

   \[
   (2.11) \quad c_\delta := \min(1/e, \exp(-\delta(M_\delta)));
   \]
which yields:

\[
\sigma^+(A) \geq \frac{(1 - c_\delta) c_\delta}{2 \delta^{-1}(\log \frac{1}{c_\delta})} = \frac{(1 - c_\delta) c_\delta \gamma(\log \frac{1}{c_\delta})}{2 \log \frac{1}{c_\delta}}.
\]

By the monotonicity of \(x\gamma(\log \frac{1}{x})\) as in 2., we conclude that:

\[
\sigma^+(A) \geq \frac{(1 - c_\delta)}{2 \log \frac{1}{c_\delta}} \sim(A) \gamma(\log \frac{1}{\sigma(A)}).
\]

(4) \(\sim(A) < c_\delta\), \(g(x) - g(0) \geq \log \frac{1}{c_\delta}\) and \(\frac{\sigma^+(A)}{\gamma(g(x) - g(0))} \geq 1/e\). Since \(\gamma\) is non-decreasing:

\[
\sigma^+(A) \geq \frac{1}{e} \gamma(g(x) - g(0)) \geq \frac{1}{ec_\delta} c_\delta \gamma(\log \frac{1}{c_\delta}).
\]

Using the monotonicity of \(x\gamma(\log \frac{1}{x})\) as in 2., we conclude that:

\[
\sigma^+(A) \geq \frac{1}{ec_\delta} \sim(A) \gamma(\log \frac{1}{\sigma(A)}).
\]

(5) \(\sim(A) < c_\delta\), \(g(x) - g(0) \geq \log \frac{1}{c_\delta}\) and \(\frac{\sigma^+(A)}{\gamma(g(x) - g(0))} < 1/e\). Recall that by (2.10):

\[
\sigma(A) \leq \frac{\sigma^+(A)}{\gamma(g(x) - g(0))} < \frac{1}{e},
\]

implying in particular that \(\sim(A) = \sigma(A)\). We will show:

\[
(2.12) \quad \sigma^+(A) \geq D_\delta \frac{\sigma^+(A)}{\gamma(g(x) - g(0))} \gamma \left( \log \frac{\gamma(g(x) - g(0))}{\sigma^+(A)} \right),
\]

which by the monotonicity of \(x\gamma(\log \frac{1}{x})\) on \([0,1/e]\) will imply:

\[
\sigma^+(A) \geq D_\delta \gamma(A) \gamma(\log \frac{1}{\sigma(A)}).
\]

(2.13)

Denote \(V_x = g(x) - g(0)\). Then (2.12) is equivalent to showing:

\[
\frac{\gamma(V_x)}{\gamma(V_x)} \left( 1 + \frac{\log \frac{\gamma(V_x)}{V_x} \exp(-g(0))}{V_x} \right) \leq 1/D_\delta.
\]

Recall that \(\gamma\) is non-decreasing and note that \(\frac{\gamma(x)}{x} = \frac{1}{\delta^{-1}(x)}\) is non-increasing. Requiring that \(D_\delta \leq 1\), it is therefore enough to show:

\[
1 + \frac{\log \frac{\gamma(V_x)}{V_x} \exp(-g(0))}{V_x} \leq 1/D_\delta.
\]

Denoting \(B_\delta := 1/D_\delta - 1\), the latter is equivalent to:

\[
\gamma(V_x) \leq \exp(B_\delta V_x) \exp(-g(0)),
\]

which from the definition of \(\gamma\) is equivalent to:

\[
\delta(V_x \exp(-B_\delta V_x) \exp(g(0))) \leq V_x.
\]
The maximum of the function $z \mapsto z \exp(-B_\delta z)$ is equal to $1/(eB_\delta)$, hence it is enough to require that:

$$\delta \left( \frac{\exp(g(0))}{eB_\delta} \right) \leq V_x.$$  

We have assumed that $V_x = g(x) - g(0) \geq \log \frac{1}{c_\delta}$; therefore by the definition (2.11) of $c_\delta$ the following condition will suffice:

(2.14) \[ \frac{\exp(g(0))}{eB_\delta} \leq M_\delta. \]

By Lemma 2.5, (2.14) holds for $B_\delta = 2/e$ (independent of $\delta$ in fact!). To conclude, (2.13) is satisfied with $D_\delta = e^{e+2}$.

Summing up all the five requirements for the constant $C_\delta$ in the conclusion of the proposition, we see that we can choose:

$$C_\delta := \frac{e - 1}{2e \max(\delta(M_\delta), 1)},$$

as claimed. \hfill \Box

2.3. A simpler proof with further assumptions. Note that the uniform convexity (1.1) of $g$ was not used in the statement and proof of Proposition 2.3. We remark here that by using this property, we obtain a simpler proof of a one-dimensional isoperimetric inequality, which may be used to complete the proof of Theorem 1.1 in place of Proposition 2.3. The key observation is the following:

Lemma 2.8. Suppose $g : (\mathbb{R}, |.|) \to \mathbb{R} \cup \{+\infty\}$ satisfies (1.1), i.e.:

(2.15) \[ \frac{g(x) + g(y)}{2} - g \left( \frac{x + y}{2} \right) \geq \delta(|x - y|) \geq 0, \quad x, y \in \mathbb{R}. \]

Then for any $x_0 \in \mathbb{R}$:

(2.16) \[ g(x) \geq g(x_0) + g'(x_0)(x - x_0) + 2\delta(|x - x_0|), \]

where $g'(x_0)$ is any value between $g'_l(x_0)$ and $g'_r(x_0)$, the left and right derivatives at $x_0$, respectively.

Proof. Immediate by applying Lemma 2.2 to the function $g - g'(x_0)(x - x_0)$, which attains its minimum at $x_0$. \hfill \Box

Proposition 2.9. Let $\sigma$ be a probability measure on $\mathbb{R}$ such that

$$d\sigma(x) = \exp(-g(x))dx,$$
where $g$ satisfies (2.15). Then

\begin{equation}
\sigma^+(A) \geq \widehat{\sigma}(A)\psi^{-1}\left(\frac{1}{2\sigma(A)}\right), \quad A \subset \mathbb{R},
\end{equation}

where

\begin{equation}
\psi(t) = t\phi(t), \quad \phi(t) = \int_{0}^{+\infty} \exp(tx - 2\delta(x))dx.
\end{equation}

Proof. As before, by a general result of Bobkov ([8, Proposition 2.1]) on extremal isoperimetric sets of log-concave densities, it is enough to verify (2.17) on sets $A$ of the form $(-\infty, x_0]$ and $[x_0, \infty)$. By symmetry, we may restrict ourselves to sets $[x_0, +\infty)$, $\sigma([x_0, \infty)) = a \leq 1/2$.

Denote $a^+ = \exp(-g(x_0))$. By (2.16),

\begin{equation}
(2.19) \quad a = \int_{x_0}^{\infty} \exp(-g(x))dx \leq a^{+}\phi(-g'(x_0)),
\end{equation}

and similarly

\begin{equation}
(2.20) \quad 1/2 \leq 1 - a \leq a^{+}\phi(g'(x_0)).
\end{equation}

Now consider two cases.

**Case 1:** $g'(x_0) > 0$. By (2.18), $\phi(-g'(x_0)) \leq 1/g'(x_0)$; hence $g'(x_0) \leq a^+/a$ using (2.19) and $a^{+}\phi(a^+/a) \geq a^{+}\phi(g'(x_0)) \geq 1/2$ using (2.20). Therefore

$$\psi(a^+/a) = (a^+/a)\phi(a^+/a) \geq 1/2a,$$

which implies (2.17).

**Case 2:** $g'(x_0) \leq 0$. By (2.20), $a^{+}\phi(0) \geq a^{+}\phi(g'(x_0)) \geq 1/2$, hence

\begin{equation}
(2.21) \quad a^+ \geq \frac{1}{2\phi(0)}.
\end{equation}

Next, since $\phi$ is monotone, $\phi\left(\frac{1}{2\phi(0)}\right) \geq \phi(0)$, hence $\psi\left(\frac{1}{2\phi(0)}\right) \geq \frac{1}{2a}$, and we conclude by (2.21) that:

$$a^+ \geq \frac{1}{2\phi(0)} \geq a\psi^{-1}\left(\frac{1}{2a}\right).$$

\[\square\]

**Remark 2.1.** It is easy to verify that the function $\phi$ defined in (2.18) is log-convex, i.e. $\log \phi$ is convex.

**Remark 2.2.** Note that when $\delta(t) = ct^p$ ($p \geq 2$), the inequalities obtained in Propositions 2.3 and 2.9 are equivalent, up to universal constants.
3. Lipschitz Maps

This section is dedicated to the proof of an extended form of Theorem 1.5.

**Proposition 3.1.** Let \( \mu \) be a finite absolutely continuous measure on \( \mathbb{R}^n \). There exists a \( \mu \)-a.e. unique radial map \( T \) that pushes \( \mu \) forward to the restriction of the Lebesgue measure to some star-shaped set \( K \subset \mathbb{R}^n \).

If \( d\mu = f \, d\text{mes}_n \), we may choose \( K = K_f \) and \( T = T_f \), where:

\[
K_f = \{ x \in \mathbb{R}^n ; v(x) \leq 1 \},
\]

\[
v(x) = \left( n \int_0^{+\infty} f(rx) r^{n-1} \, dr \right)^{-\frac{1}{n}},
\]

and \( T_f \) is given by \( T_f(0) = 0 \) and:

\[
T_f(x) = \left( \frac{ \int_0^1 f(rx) r^{n-1} \, dr }{ \int_0^\infty f(rx) r^{n-1} \, dr } \right)^{\frac{1}{n}} \frac{x}{v(x)}, \quad x \neq 0.
\]

**Proof of Proposition 3.1.** Let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be a radial map pushing \( \mu \) forward to the Lebesgue measure restricted to a star-shaped body \( K \). Define:

\[
w(x) = \inf \{ t > 0 ; t^{-1}x \in K \};
\]

then the restriction of \( T \) to a ray \( \mathbb{R}_+x \), \( w(x) = 1 \), has the form:

\[
xr \mapsto u(x,r)x, \quad r > 0.
\]

Passing to polar coordinates and using the Fubini theorem, we see that \( T_\ast \mu \) is equal to the restriction of \( \text{mes}_n \) to \( K \) iff, for almost every ray \( \mathbb{R}_+x \), \( w(x) = 1 \), the map \( u(x,\cdot) \) pushes \( f(rx)r^{n-1}dr \) forward to \( 1_{(0,1]}r^{n-1}dr \); that is, if

\[
\int_0^1 \phi(r)r^{n-1}dr = \int_0^\infty \phi(u(x,r))f(rx)r^{n-1}dr
\]

for any test function \( \phi \in C_0(\mathbb{R}_+) \). Setting \( \phi = 1_{(0,T]} \) in (3.3) and letting \( T \to \infty \), we see that

\[
\frac{1}{n} = \int_0^\infty f(rx)r^{n-1}dr.
\]

Hence \( v(x) = 1 \) for (almost) every \( x \) such that \( w(x) = 1 \). Both \( v \) and \( w \) are homogeneous functions, hence \( v(x) = w(x) \) for \( \mu \)-a.e. \( x \in \mathbb{R}^n \).

Now use \( \phi = 1_{[0,u(x,s)]} \) in (3.3). Since \( u(x,\cdot) \) is monotone, we deduce:

\[
u(x,s)^n = n \int_0^s f(rx)r^{n-1}dr = \frac{\int_0^s f(rx)r^{n-1}dr}{\int_0^\infty f(rx)r^{n-1}dr},
\]

at every point of continuity \( s \) of \( u(x,\cdot) \). Therefore

\[
T(sx) = \left( \frac{\int_0^s f(rx)r^{n-1}dr}{\int_0^\infty f(rx)r^{n-1}dr} \right)^{1/n} x, \quad v(x) = 1,
\]

which is equivalent to (3.2).

\( \square \)

**Remark 3.1.** Note that in particular, \( \text{mes}_n(K_f) = \text{mes}_n(K) = \mu(\mathbb{R}^n) \).
The following proposition was proved by K. Ball [5] for even log-concave functions and extended by Klartag [25, Theorem 2.2] to general log-concave functions.

**Proposition** (Ball). *If \( f \) is a log-concave function on \( \mathbb{R}^n \), then \( K_f \) is a convex body.*

Note that we do not assume at this stage that \( f \) is even. Therefore \( K_f \) may not necessarily be symmetric about the origin, so formally we can not identify it with the unit-ball of some norm \( \|\cdot\|_{K_f} \). Nevertheless, we denote:

\[
\|x\|_{K_f} = \left( n \int_0^{\infty} f(rx)r^{n-1}dr \right)^{-\frac{1}{n}};
\]

by the above proposition, this is a convex function on \( \mathbb{R}^n \), which is in addition homogeneous. By definition (3.1), we have:

\[
K_f = \left\{ x \in \mathbb{R}^n; \|x\|_{K_f} \leq 1 \right\}.
\]

In addition, we denote:

\[
\widehat{K}_f = K_f \cap -K_f
\]

which is now a convex body symmetric about the origin, and we associate with it the corresponding norm \( \|\cdot\|_{\widehat{K}_f} \).

We can now state the following result, which extends Theorem 1.5:

**Theorem 3.2.** Let \( f \) denote a log-concave function on \( \mathbb{R}^n \) with barycenter at the origin such that \( 0 < \int f(x)dx < \infty \). Let \( \mu \) denote the measure with density \( f \), and let \( \lambda \) denote the restriction of the Lebesgue measure to \( K_f \). Denote by \( T = T_f \) the canonical radial map (given by (3.2)) such that \( T_\ast \mu = \lambda \), and let \( u : (\mathbb{R}^n, \|\cdot\|_{\widehat{K}_f}) \to [0, 1] \) be defined by:

\[
T(x) = u(x) \frac{x}{\|x\|_{K_f}}
\]

for \( x \neq 0 \) and \( u(0) = 0 \). Then \( \|u\|_{Lip} \leq Cf(0)^{1/n} \), where \( C > 0 \) is a universal constant.

When \( f \) is in addition even, \( \widehat{K}_f = K_f \) and \( \|\cdot\|_{K_f} \) is indeed a norm. Theorem 1.5 is then deduced from Theorem 3.2 using the following lemma, which was essentially proved by Bobkov and Ledoux [12].

**Lemma 3.3.** Let \( V = (X, \|\cdot\|) \) denote a normed space, and let \( T : V \to V \) be the map defined by \( T(0) = 0 \) and:

\[
T(x) = u(x) \frac{x}{\|x\|}
\]

for \( x \neq 0 \), where \( u : X \to \mathbb{R}_+ \) has a finite Lipschitz constant and satisfies \( u(0) = 0 \). Then:

\[
\|T\|_{Lip} \leq 3 \|u\|_{Lip}.
\]
Proof. Let \( x, y \in X \). By continuity, we may assume that \( x, y \neq 0 \). Then:

\[
\|T(x) - T(y)\| = \left\| u(x) \frac{x}{\|x\|} - u(y) \frac{y}{\|y\|} \right\| \\
\leq \left\| u(x) \frac{x}{\|x\|} - u(x) \frac{y}{\|y\|} \right\| + \left\| u(x) \frac{y}{\|y\|} - u(y) \frac{y}{\|y\|} \right\| \\
= |u(x) - u(0)| \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| + |u(x) - u(y)| \\
\leq \|u\|_{\text{Lip}} \|x\| \left( \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| + \left\| \frac{y}{\|x\|} - \frac{y}{\|y\|} \right\| \right) + \|u\|_{\text{Lip}} \|x - y\| \\
= \|u\|_{\text{Lip}} \|x\| \|y\| \left( \frac{1}{\|x\|} - \frac{1}{\|y\|} \right) + 2 \|u\|_{\text{Lip}} \|x - y\| \leq 3 \|u\|_{\text{Lip}} \|x - y\|.
\]

\[\square\]

For the proof of Theorem 3.2, we need to compile several known results about log-concave functions.

3.1. Additional Preliminaries. Another convex body associated to a log-concave function \( f \) on \( \mathbb{R}^n \) was put forth by B. Klartag and V. Milman [26]. Assume that \( f(0) > 0 \), we define the (convex) body \( K^0_f \) as the set:

\[
K^0_f = \{ x \in \mathbb{R}^n; f(x) \geq f(0) \exp(-n) \}.
\]

We will use a relation between \( K_f \) and \( K^0_f \) that was proved (under slightly different assumptions) by Klartag and Milman [26, Lemmata 2.1, 2.2]:

**Proposition 3.4** (Klartag–Milman). Let \( f \) be a log-concave density on \( \mathbb{R}^n \), and assume that \( f(0) > 0 \). Then:

\[
K_f \subset C_n (\sup_x f(x))^\frac{n}{n+1} K^0_f,
\]

where \( C_n > 1 \) and \( C_n \to 1 \) as \( n \to \infty \). Moreover, if \( f \) attains its maximum at 0, then:

\[
f(0)^\frac{1}{2} K^0_f \subset D_n K_f,
\]

where \( D_n > 2 \) and \( D_n \to 2 \) as \( n \to \infty \).

The next lemma is a one dimensional computation for log-concave functions. For even functions, this fact goes back to Ball [4], and Milman and Pajor [30]. For arbitrary log-concave functions, this was extended by Klartag [25, Lemma 2.6] as follows:

**Lemma 3.5.** Let \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) denote a non-constant log-concave function, and let \( n \geq 1 \). Assume that \( f(0) = 1 \) and that:

\[
\sup_x f(x) \leq \exp(n).
\]

Then:

\[
C_1 \leq \frac{n^{n+1}}{e(n+1)} \leq \frac{\int_0^\infty f(r)r^n dr}{(\int_0^\infty f(r)r^{n-1} dr)^{\frac{n+1}{n}}} \leq \frac{n!}{(n-1)!^{\frac{n+1}{n}}} \leq C_2,
\]

where \( C_1, C_2 > 0 \) are universal constants. In fact, the assumption (3.8) is not needed for the right-hand side of the inequality.
The last proposition we need is due to M. Fradelizi [20, Theorem 4]:

**Proposition 3.6** (Fradelizi). Let $f$ denote a log-concave density on $\mathbb{R}^n$ such that $0 < \int f(x)dx < +\infty$, and let $x_0$ denote its barycenter. Then:

$$g(x_0) \geq \exp(-n) \sup_{x \in \mathbb{R}^n} g(x).$$

3.2. **Proof of Theorem 3.2.** By (3.2), $T(x) = u(x) \frac{x}{\|x\|_{K_f}}$ for $x \neq 0$, where $u$ is given by:

$$u(x) = \left( \frac{\int_0^1 r^{n-1} f(rx)dr}{\int_0^\infty r^{n-1} f(rx)dr} \right)^\frac{1}{n}$$

for $x \neq 0$ and $u(0) = 0$. We thus verify that $u$ is continuous at 0.

**Step 1:** Reduction to smooth $f$.

Define, for $\varepsilon > 0$, $f_\varepsilon := f \ast \varepsilon^{-n}G(x/\varepsilon)$, where $G$ is the standard Gaussian density on $\mathbb{R}^n$ and $\ast$ denotes convolution. Clearly $f_\varepsilon$ is a smooth function with barycenter at 0. By the Prekopa-Leindler Theorem, $f_\varepsilon$ is log-concave, as the convolution of two log-concave functions.

Let $\mu_\varepsilon$ denote the measure with density $f_\varepsilon$, $\lambda_\varepsilon$ the Lebesgue measure on $K_{f_\varepsilon}$, and let $T_\varepsilon$ denote the map radially pushing forward the measure $\mu_\varepsilon$ onto $\lambda_\varepsilon$. Let $u_\varepsilon$ be defined by

$$T_\varepsilon(x) = u_\varepsilon(x) \frac{x}{\|x\|_{K_{f_\varepsilon}}} ,$$

with $u_\varepsilon(0) = 0$. Given $x, y \in \mathbb{R}^n$, it is clear from (3.9) and (3.6) that $u_\varepsilon(x) \to u(x)$, $u_\varepsilon(y) \to u(y)$, $\|x - y\|_{K_{f_\varepsilon}} \to \|x - y\|_{K_f}$ and $\|x - y\|_{K_{f_\varepsilon}} \to \|x - y\|_{K_f}$ as $\varepsilon$ tends to 0. If we assume that $\|u_\varepsilon\|_{Lip} \leq Cf_\varepsilon(0)^{1/n}$, we have:

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq C f_\varepsilon(0)^{1/n} \|x - y\|_{K_{f_\varepsilon}} .$$

Passing to the limit as $\varepsilon \to 0$, it follows that:

$$|u(x) - u(y)| \leq C f(0)^{1/n} \|x - y\|_{K_f} ,$$

and we conclude that $\|u\|_{Lip} \leq C f(0)^{1/n}$. It is therefore enough to restrict our discussion to smooth functions.

**Step 2:** Proof for smooth functions with $f(0) = 1$.

Assume that $f(0) = 1$.

Note that since $f$ and thus $u$ are assumed to be smooth,

$$\|u\|_{Lip} = \sup_{x \in \mathbb{R}^n} \|\nabla u(x)\|_{K_f}^* ,$$

where $\|\cdot\|_{K_f}^*$ is the dual norm to $\|\cdot\|_{K_f}$.

Fixing $x \in \mathbb{R}^n$, $x \neq 0$, we will show that $\|\nabla u(x)\|_{K_f}^* \leq C$ for some universal constant $C > 0$. Write $f = \exp(-g)$, and denote for short:

$$A = \int_0^1 r^{n-1} f(rx)dr$$

and

$$B = \int_1^\infty r^{n-1} f(rx)dr .$$
note that
\[ \nabla A = - \int_0^1 r^n f(rx) \nabla g(rx) dr \quad \text{and} \quad \nabla B = - \int_1^\infty r^n f(rx) \nabla g(rx) dr . \]

By Proposition 3.6, since \( f(0) = 1 \) and 0 is the barycenter of \( f \), then \( \sup_x f(x) \leq \exp(n) \).

This clearly implies that \( A \leq \exp(n)/n \), and that \( g(x) \geq -n \). Denote also:
\[ A^* = \int_0^1 r^n f(rx) \|\nabla g(rx)\|_{K_f}^r dr \quad \text{and} \quad B^* = \int_1^\infty r^n f(rx) \|\nabla g(rx)\|_{K_f}^r dr . \]

Then by (3.9)
\[ \|\nabla u(x)\|_{\overline{K}_f}^r = \frac{1}{n} \left( \frac{A}{A + B} \right) \frac{1}{\frac{1}{n} |\nabla A(A + B) - A(\nabla A + \nabla B)|_{K_f}^r} \]
\[ \leq \frac{1}{n} \left( \frac{A}{A + B} \right) \frac{1}{\frac{1}{n} A^* B + A B^*} \]
\[ \leq \frac{1}{n} \frac{A^*}{A} + \frac{1}{n} \left( \frac{A}{A + B} \right) \frac{1}{\frac{1}{n} A^* + B^*} \]
\[ \leq \frac{1}{n} \frac{A^*}{A} + \frac{1}{n} \frac{A^* + B^*}{(A + B)^{\frac{n+1}{n}}} . \]

Note that by the convexity of \( g \), for all \( x, y \in \mathbb{R}^n \):
\[ g(y) \geq g(x) + \langle \nabla g(x), y - x \rangle . \]

Recall the definition (3.7), stating that \( y \in K^0_f \) iff \( g(y) \leq n + g(0) = n \), and also recall that \( g(\cdot) \geq -n \). This implies that for \( y \in K^0_f \):
\[ \langle \nabla g(x), y \rangle \leq \langle \nabla g(x), x \rangle + g(y) - g(x) \leq \langle \nabla g(x), x \rangle + 2n. \]

By Proposition 3.4 \( K_f \subset K^0_f \subset DK^0_f \), where \( D = C(\sup_x f(x))^{1/n} \leq C e \) for some universal \( C > 1 \); hence
\[ \|\nabla g(x)\|_{\overline{K}_f}^r \leq D(\langle \nabla g(x), x \rangle + 2n). \]

We will use this rough estimate to bound \( A^* \) and \( B^* \) from above. More generally, for \( 0 \leq a < b \leq \infty \),
\[ \frac{1}{D} \int_a^b r^n f(rx) \|\nabla g(rx)\|_{\overline{K}_f}^r dr \leq \int_a^b r^n f(rx)(\langle \nabla g(rx), rx \rangle + 2n) dr \]
\[ = \frac{d}{dt}\bigg|_{t=1} \left( - \int_a^b r^n f(trx) dr \right) + 2n \int_a^b r^n f(rx) dr \]
\[ = \frac{d}{dt}\bigg|_{t=1} \left( t^{-(n+1)} \int_{at}^{bt} r^n f(rx) dr \right) + 2n \int_a^b r^n f(rx) dr \]
\[ = (3n + 1) \int_a^b r^n f(rx) dr + a^{n+1} f(ax) - b^{n+1} f(bx) . \]

Of course the last term is interpreted as 0 when \( b = \infty \). With this bound in mind, let:
\[ A' = \int_0^1 r^n f(rx) dr \quad \text{and} \quad B' = \int_1^\infty r^n f(rx) dr . \]
Applying (3.11), we see that:

\[
\frac{A^*}{D} \leq (3n + 1)A' - f(x) ;
\]
\[
\frac{(A^* + B^*)}{D} \leq (3n + 1)(A' + B') ,
\]

Hence by (3.10)

\[
\|\nabla u(x)\|_{K_f}^* \leq \frac{(3n + 1)D}{n} \left( \frac{A'}{A} + \frac{A' + B'}{A + B} \right)^{\frac{n+1}{n}}.
\]

Obviously \(A' \leq A\) since \(r \leq 1\) in the integrand of \(A'\). By Lemma 3.5 (that is applicable since \(f(0) = 1\)) we have:

\[
A' + B' = \int_0^\infty r^n f(xr) dr \leq C \left( \int_0^\infty r^{n-1} f(xr) dr \right)^{\frac{n+1}{n}} = C(A + B)^{\frac{n+1}{n}},
\]

where \(C > 0\) is some universal constant. It follows that:

\[
\|u\|_{Lip} = \sup_{x \in \mathbb{R}^n} \|\nabla u(x)\|_{K_f}^* \leq 4D(1 + eC).
\]

**Step 3:** Proof for general smooth functions.

We have shown the assertion of the theorem for smooth functions \(f\) with \(f(0) = 1\). In the general case, obviously \(f(0) > 0\), since the barycenter of the log-concave \(f\) is at the origin. Let us push forward \(f(x)dx\) by the map \(S(x) = f(0)^{1/n}x\) to obtain \(f'(x)dx\), where:

\[
f'(x) = f(0)^{-1} f(f(0)^{-1/n}x).
\]

Clearly \(K_{f'}\) is a homothetic copy of \(K_f\), and since

\[
\text{mes}_n(K_{f'}) = \int f'(x) dx = \int f(x) dx = \text{mes}_n(K_f),
\]

we see that \(K_{f'} = K_f\). Let \(T\) denote the radial map pushing forward \(f'(x)dx\) to the restriction of the Lebesgue measure on \(K_f\), denoted \(\lambda\). Let \(u' : (\mathbb{R}^n, \|\cdot\|_{K_f}) \to [0, 1]\) be defined by:

\[
T'(x) = u'(x) \frac{x}{\|x\|_{K_f}},
\]

and \(u'(0) = 0\). Since \(f'(0) = 1\) and \(f'\) is smooth, step 2 implies that \(\|u'\|_{Lip} \leq C\). Obviously \(T = T' \circ S\) (e.g. by uniqueness of the radial map pushing forward \(f(x)dx\) onto \(\lambda\), and hence \(u = u' \circ S\). This implies:

\[
\|u\|_{Lip} = \|u'\|_{Lip} f(0)^{1/n} \leq C f(0)^{1/n},
\]

and concludes the proof.  \(\square\)

**Remark 3.2.** Of course the proof uses the fact that the barycenter of \(f\) is at the origin in a very indirect way. In fact, it is clear from the proof that we may use any log-concave function \(f\) for which:

\[
f(0) \geq D^{-n} \sup_{x \in \mathbb{R}^n} f(x),
\]

for some \(D \geq 1\), yielding \(\|u\|_{Lip} \leq C(D) f(0)^{1/n}\), where \(C(D)\) is a constant depending on \(D\).
As an immediate corollary of Theorem 1.5, we obtain the Bobkov-Ledoux Proposition from the introduction, although the direct route taken by Bobkov and Ledoux in [12] is simpler in this case and recovers a better universal constant in the bound.

**Proof of the Bobkov–Ledoux Proposition.** It is easy to see that the Lipschitz constant of $S$ as a map acting on $(\mathbb{R}^n, \|\cdot\|)$ is invariant to scaling of the Lebesgue measure, so we may assume that $\text{mes}_n(K) = 1$. By Theorem 1.5,

$$\|S\|_{\text{Lip}} \leq C f(0)^{1/n} = C T(1 + n/p)^{-1/n}.$$ 

□

We will see in the next section how Theorem 1.5 may be used to transfer isoperimetric inequalities from log-concave measures to uniform measures on convex bodies.

### 4. General Uniformly Convex Bodies

In this section we give a proof of Proposition 1.4 and provide the details that lead to Theorem 1.6.

Let $\delta = \delta_V$ denote the modulus of convexity of a normed space $V = (X, \|\cdot\|)$. It is known that $\delta$ is not necessarily a convex function; we denote by $\tilde{\delta}$ the maximal convex function majorated by $\delta$. We summarise several known facts about $\delta$ and $\tilde{\delta}$ (see Lindenstrauss and Tzafriri [27, Proposition 1.6.1, Lemmata 1.6.7, 1.6.8]).

**Lemma 4.1.**

1. $\delta(t)/t$ is non-decreasing on $[0, 2]$.
2. $\delta(t/2) \leq \tilde{\delta}(t) \leq \delta(t)$ for all $t \in [0, 2]$.
3. There exists a constant $C \geq 1$ such that $\tilde{\delta}(t)/s^2 \leq C \tilde{\delta}(s)/s^2$, for all $0 \leq t \leq s \leq 2$.

The following crucial fact is due to Figiel and Pisier [19] (see also [27, Lemma 1.6.10]):

**Proposition 4.2 (Figiel–Pisier).** Let $x, y \in X$ such that $\|x\|^2 + \|y\|^2 \leq 2$. Then:

$$\|x + y\|^2 \leq 4 - 4\delta(\|x - y\|/2).$$

Proposition 1.4 is an easy corollary of these lemmata.

**Proof of Proposition 1.4.** Let $x, y \in X$ such that $\|x\|^2 + \|y\|^2 \leq 2$, and denote $s^2 := (\|x\|^2 + \|y\|^2)/2 \leq 1$. If $s = 0$ then $\|x\| = \|y\| = 0$ and the claim is trivial. Otherwise, denote $x' = x/s$ and $y' = y/s$, so that $\|x'\|^2 + \|y'\|^2 = 2$. Hence by Proposition 4.2:

$$\left\|\frac{x' + y'}{2}\right\|^2 \leq 1 - \delta \left(\frac{\|x' - y'\|}{2}\right),$$

or equivalently:

$$\left\|\frac{x + y}{2}\right\|^2 \leq s^2 - s^2 \delta \left(\frac{\|x' - y'\|}{2s}\right).$$

Now, $s \leq 1$; hence by Lemma 4.1 we have for any $t \in [0, 2s]$:

$$s^2 \delta(t/s) \geq s^2 \tilde{\delta}(t/s) \geq c\delta(t) \geq c\delta(t/2),$$
where \( c > 0 \) is a universal constant. Applying this for 
\[
    t = \frac{\|x - y\|}{2} \leq \frac{\|x\| + \|y\|}{2} \leq s, 
\]
we conclude that:
\[
    \left\| \frac{x + y}{2} \right\|^2 \leq \frac{\|x\|^2 + \|y\|^2}{2} - c\delta \left( \frac{\|x - y\|}{4} \right),
\]
as required. \( \square \)

Now we can fill the details in the proof of Theorem 1.6. Assume that 
\( V = (\mathbb{R}^n, \|\cdot\|) \) is 
a uniformly convex space, and let \( \delta = \delta_V \) denote its modulus of convexity as before. Scale 
the Lebesgue measure on \( \mathbb{R}^n \) so that \( \text{mes}_n \{\|x\| \leq 1\} = 1 \), since the statement of Theorem 
1.6 is invariant to this scaling. Now denote by \( \mu \) the probability measure with density:
\[
    f(x) = \frac{1}{Z} \exp(-n/c\|x\|^2) 1(\|x\| \leq 1/4)
\]
with respect to the Lebesgue measure, where \( c > 0 \) is the constant from Proposition 1.4. Here \( Z > 0 \) is a scaling factor so that \( \mu \) be indeed a probability measure. Integrating on 
level sets of \( \|\cdot\| \), it is clear that:
\[
    Z = \int_{\mathbb{R}^n} \exp \left( -\frac{n}{c} \|4x\|^2 \right) 1(\|x\| \leq 1/4) \, dx 
\]
\[
    = n \int_0^{1/4} \exp \left( -\frac{16}{c} ns^2 \right) s^{n-1} \, ds,
\]
and in particular \( Z^{1/n} \geq c' > 0 \).

Write \( f = \exp(-g) \), with \( g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \). Proposition 1.4 then implies that \( g \) is 
uniformly convex, and satisfies:
\[
    \frac{g(x) + g(y)}{2} - g \left( \frac{x + y}{2} \right) \geq n\delta_1(\|x - y\|),
\]
where \( \delta_1 \) coincides with \( \delta \) on \( [0, 1/4] \) and \( \delta_1(t) = +\infty \) for \( t > 1/4 \). Since \( \delta(t)/t \) is non-
decreasing by Lemma 4.1, so is \( \delta_1(t)/t \), and assumption (1.2) is fulfilled. We can therefore 
apply Theorem 1.1, and deduce an isoperimetric inequality for \( \mu \) on \( V \):
\[
    \mu^+(\|\cdot\|)(A) \geq C_{n,\delta} \widehat{\mu}(A) \gamma_n \left( \log \frac{1}{\mu(A)} \right) \quad \text{for all } A \subset \mathbb{R}^n,
\]
where \( C_{n,\delta} \) is given by (1.15) and:
\[
    \gamma_n(t) = \frac{t}{\delta_1^{-1}(t/(2n))}.
\]

We would now like to transfer this isoperimetric inequality to \( \lambda_V \), the uniform probability 
measure on \( K_V = \{\|x\| \leq 1\} \), via a radial Lipschitz map. Clearly, \( K_f \) is a homothetic copy 
of \( K_V \), and since
\[
    \text{mes}_n(K_f) = \int f(x) \, dx = 1 = \text{mes}_n(K_V),
\]
it follows that \( K_f = K_V \). Note also that
\[
    f(0)^{1/n} = Z^{-1/n} \leq (c')^{-1}.
\]
Applying Theorem 1.5, it follows that the Lipschitz constant of the radial map pushing forward \( \mu \) onto \( \lambda _{V} \) is bounded by a universal constant. This implies the statement of Theorem 1.6.

5. Concentration and functional inequalities

5.1. Concentration of measure on uniformly convex bodies. In this subsection, we discuss the connection between our results and the following Gromov–Milman inequality [22], that we cite in the form of Arias-de-Reyna, Ball, and Villa [1].

**Theorem** (Gromov–Milman). Let \( V = (\mathbb{R}^n, \| \cdot \|) \) be a normed space; let \( \delta = \delta _{V} \) be its modulus of convexity, and let \( \lambda \) be the uniform measure on the unit ball of \( V \). Then

\[
1 - \lambda (B_{\varepsilon, \| \cdot \|}) \leq \frac{1}{\lambda (B)} \exp (-2n\delta (\varepsilon)) \quad \text{for all } B \subset \mathbb{R}^n.
\]

In particular, if \( \delta (\varepsilon) \geq \alpha '\varepsilon ^p \), then

\[
1 - \lambda (B_{\varepsilon, \| \cdot \|}) \leq \frac{1}{\lambda (B)} \exp (-2\alpha 'n\varepsilon ^p) \quad \text{for all } B \subset \mathbb{R}^n.
\]

Let us compare this to our results. First assume \( \delta (\varepsilon) \geq \alpha '\varepsilon ^p \); then (1.10) holds with \( \alpha = \alpha '/2p \) (as mentioned in Subsection 1.2.2). Therefore by Theorem 1.3

\[
\lambda _{\| \cdot \|}^+(A) \geq C' \left( \frac{1}{\lambda (A)} \right) ^{1/p} \frac{1}{\lambda (A)} \log \left( \frac{1}{\lambda (A)} \right) \quad \text{for all } B \subset \mathbb{R}^n,
\]

where \( C' \) is a universal constant. Hence by Corollary 1.8

\[
1 - \lambda (B_{\varepsilon, \| \cdot \|}) \leq \exp \left\{ - \left[ \log ^{1/p} \frac{1}{1 - \lambda (B)} + \frac{C(\alpha ')^{1/p}n^{1/p}\varepsilon}{p} \right]^p \right\}
\]

The right-hand side in (5.4) is at most

\[
(1 - \lambda (B)) \exp \left\{ -C(\alpha ')^{1/p}n^{1/p} \log ^{1/p} \frac{1}{1 - \lambda (B)} \varepsilon \right\} < 1 - \lambda (B);
\]

hence (5.4) yields a meaningful bound for any \( \varepsilon > 0 \), whereas (5.2) is meaningful for

\[
\varepsilon \geq \left\{ \frac{1}{2\alpha 'n} \log \frac{1}{\lambda (B)(1 - \lambda (B))} \right\} ^{1/p}.
\]

On the other hand, for larger \( \varepsilon \) the right-hand side of (5.4) behaves like

\[
\exp \left\{ - \frac{C'p}{p ' \alpha 'n^p \varepsilon ^p} \right\};
\]

that is, we lose a factor \( p ' \) in the exponent.

The preceding discussion can be extended to arbitrary moduli of convexity. In the general case, Theorem 1.6 yields

\[
\lambda _{\| \cdot \|}^+(A) \geq C_{n, \delta} \left( \frac{\lambda (A)}{\lambda (A)} \right) \frac{1}{\delta ^{-1} \left( \frac{1}{2n} \log \frac{1}{\lambda (A)} \right)};
\]
hence by Proposition 1.7

\[ (5.6) \quad 1 - \lambda(B_{\varepsilon, \|\cdot\|}) \leq \exp \left\{ -h^{-1}_a(B)(\varepsilon) \right\}, \]

where

\[ h_a(x) = \int_{\log 1/a}^{x} \frac{\delta^{-1}(y/2n)dy}{C_{n,\delta}^x y}. \]

By Lemma 4.1 we can assume without loss of generality that \( \delta \) is convex (and \( \delta^{-1} \) is concave). Then,

\[ h_a(x) = \int_{\log 1/a}^{x} \frac{\delta^{-1}(y/2n)dy}{C_{n,\delta}^x y} \]
\[ \leq \int_{\log 1/a}^{x} \frac{dy}{C_{n,\delta}^x y} \delta^{-1} \left\{ \frac{1}{2n} \int_{\log 1/a}^{x} \frac{dy}{C_{n,\delta}^x y} \right\} \]
\[ = \log x - \log \log 1/a \delta^{-1} \left\{ \frac{1}{2n} \frac{x - \log 1/a}{\log x - \log \log 1/a} \right\}. \]

Now, \( t \mapsto \delta^{-1}(t)/t \) is decreasing, hence

\[ (5.7) \quad h_a(x) \leq \frac{1}{C_{n,\delta}'} \delta^{-1} \left\{ \frac{1}{2n} (x - \log 1/a) \right\} \quad \text{if } x \leq e \log 1/a. \]

On the other hand,

\[ h_a(e \log 1/a) = \int_{\log 1/a}^{e \log 1/a} \frac{\delta^{-1}(y/2n)dy}{C_{n,\delta}^x y} \geq \frac{e - 1}{e C_{n,\delta}'} \delta^{-1} \left\{ \frac{e \log 1/a}{2n} \right\}; \]

hence for \( \varepsilon \leq \frac{e - 1}{e C_{n,\delta}'} \delta^{-1} \left\{ \frac{e \log 1/a}{2n} \right\} \), \( x = h_a^{-1}(\varepsilon) \leq e \log 1/a \), and (5.7) implies:

\[ h_a^{-1}(\varepsilon) \geq 2n\delta(C_{n,\delta}'\varepsilon) + \log 1/a. \]

We conclude by (5.6) that:

\[ (5.8) \quad 1 - \lambda(B_{\varepsilon, \|\cdot\|}) \leq (1 - \lambda(B)) \exp \left\{ -2n\delta(C_{n,\delta}'\varepsilon) \right\}. \]

Again, (5.8) is better than (5.1) for small \( \varepsilon \); if

\[ \varepsilon \leq \frac{e - 1}{e C_{n,\delta}'} \delta^{-1} \left\{ \frac{e \log 1/a}{2n} \right\} \]

the inequalities (5.1) and (5.8) are similar, whereas for larger \( \varepsilon \) an inequality of type (5.8) can only be deduced from (5.5) under additional regularity assumptions on \( \delta \).
5.2. **Proofs.** It remains to prove Propositions 1.7 and 1.9.

**Proof of Proposition 1.7.** Let $B \subset \mathbb{R}^n$ be a Borel set such that:

$$a = 1 - \mu(B) \leq 1/2;$$

the proof easily extends to the complementary case $a > 1/2$.

Denote $f(t) = 1 - \mu(B_t)$. Our assumptions then read:

$$f(0) = a; \quad df/dt(t) \leq -f(t)\gamma(-\log f(t))$$

(where strictly speaking $df/dt$ should be the upper left derivative). Setting $g = -\log f$, 

$$g(0) = \log 1/a; \quad dg/dt \geq \gamma \circ g,$$

and if $h = g^{-1}$, 

$$h(\log 1/a) = 0 \quad \text{and} \quad dh/dt \leq 1/(\gamma).$$

Therefore

$$h(x) \leq \int_{\log 1/a}^{x} \frac{dy}{\gamma(y)} = h_a(x),$$

and

$$f(t) = \exp(-h^{-1}(t)) \leq \exp(-h_a^{-1}(t)),$$

as required.

The converse direction is obvious. \(\square\)

**Proof of Proposition 1.9.** Let us show that 1. implies 2. Let $F$ be a function satisfying (1.22); assume for simplicity that the distribution of $F$ has no atoms except for 0 and 1 and that $\mu\{F = 0\} = 1/2, \mu\{F = 1\} = t = 1/2^k$. Choose

$$0 = u_1 < u_2 < \cdots < u_k = 1$$

so that

$$\mu\{u_i < F < u_{i+1}\} = 1/2^{i+1}. $$

Then

$$\int \|\nabla F\|^q d\mu = \sum \int_{u_i < F \leq u_{i+1}} \|\nabla F\|^q d\mu$$

$$\geq \sum \frac{1}{2^{i+1}} \left\{ 2^{i+1} \int_{u_i < F < u_{i+1}} \|\nabla F\|^q d\mu \right\}^q$$

by Jensen’s inequality. Now, apply 1. to the function

$$F_i = \max \left( 0, \min \left( 1, \frac{F - u_i}{u_{i+1} - u_i} \right) \right).$$

Since $\mu\{F_i = 1\} = \mu\{F \geq u_i + 1\} = 1/2^{i+1}$, we obtain:

$$\int_{u_i < F < u_{i+1}} \|\nabla F\|^q d\mu \geq c \log^{1/q} 2^{i+1} \frac{2^{i+1}}{2^{i+1}};$$
therefore
\[
\int \|\nabla F\|_q^q d\mu \geq \sum_{i=1}^k \left( c c_0 (u_{i+1} - u_i) \log^{1/q} 2^{i+1} \right)^q
\]
\[
\geq c'' c_0^q \sum (u_{i+1} - u_i)^{q (i + 1) / 2^{i+1}}
\]
\[
\geq c'' c_0^q \left( \sum (u_{i+1} - u_i)^q \right) \int \left[ \sum \left( \frac{2^{i+1}}{i+1} \right)^{p/q} \right]^{q/p}
\]
according to Hölder’s inequality. Finally,
\[
\sum_{i=1}^k \left( \frac{2^{i+1}}{i+1} \right)^{p/q} \leq C(2^k / k)^{p/q}
\]
and thence
\[
\int \|\nabla F\|_q^q d\mu \geq c'' c_0^q k \frac{k^{p/q}}{2^k} \geq c' c_0^q \log 1/t .
\]

\[
\square
\]

References
23. O. Hanner, On the uniform convexity of $L^p$ and $l^p$, Ark. Mat. 3 (1956), 239–244.

E-mail address: emanuel.milman@weizmann.ac.il

DEPARTMENT OF MATHEMATICS, THE WEIZMANN INSTITUTE OF SCIENCE, REHOVOT 76100, ISRAEL.

E-mail address: sodinale@tau.ac.il

SCHOOL OF MATHEMATICS, RAYMOND AND BEVERLY SACKLER FACULTY OF EXACT SCIENCES, TEL AVIV UNIVERSITY, TEL AVIV, 69978, ISRAEL.
CHAPTER 8

A REMARK ON TWO DUALITY RELATIONS

EMANUEL MILMAN

Integral Equations and Operator Theory 57 (2), 217-228, 2007

Abstract. We remark that an easy combination of two known results yields a positive answer, up to \( \log(n) \) terms, to a duality conjecture that goes back to Pietsch. In particular, we show that for any two symmetric convex bodies \( K, T \) in \( \mathbb{R}^n \), denoting by \( N(K, T) \) the minimal number of translates of \( T \) needed to cover \( K \), one has:

\[
N(K,T) \leq N(T^\circ, (C \log(1 + n))^{-1} K^\circ C \log(1+n) \log \log(2+n)),
\]

where \( K^\circ, T^\circ \) are the polar bodies to \( K, T \), respectively, and \( C \geq 1 \) is a universal constant.

As a corollary, we observe a new duality result (up to \( \log(n) \) terms) for Talagrand’s \( \gamma_p \) functionals.

1. Introduction

Let \( K \) and \( T \) denote two convex bodies in \( \mathbb{R}^n \) (i.e. convex compact sets with non-empty interior). Throughout this note we assume that all bodies in question are centrally symmetric w.r.t. to the origin (e.g. \( K = -K \)). For a convex body \( L \), we denote by \( L^\circ \) its polar body, defined as \( L^\circ = \{ x \in \mathbb{R}^n; \langle x, y \rangle \leq 1 \ \forall y \in L \} \). The covering number of \( K \) by \( T \), denoted \( N(K,T) \), is defined as the minimal number of translates of \( T \) needed to cover \( K \), i.e.:

\[
N(K,T) = \min \left\{ N ; \exists x_1, \ldots, x_N \in \mathbb{R}^n, K \subset \bigcup_{1 \leq i \leq N} (x_i + T) \right\}.
\]

In this note, we address the following conjecture of Pietsch ([13, p. 38]) from 1972, originally formulated in operator-theoretic notations:

**Duality Conjecture for Covering Numbers.** Do there exist numerical constants \( a, b \geq 1 \) such that for any dimension \( n \) and for any two symmetric convex bodies \( K, T \) in \( \mathbb{R}^n \) one has:

\[
b^{-1} \log N(T^\circ, aK^\circ) \leq \log N(K,T) \leq b \log N(T^\circ, a^{-1}K^\circ) ?
\]

This problem may be equivalently formulated using the notion of *entropy numbers*. For a real number \( k \geq 0 \), denote the \( k \)'th entropy number of \( K \) w.r.t. \( T \) as:

\[
e_k(K,T) = \inf\{ \varepsilon > 0; N(K, \varepsilon T) \leq 2^k \}.
\]

Supported in part by BSF and ISF.
Then the duality conjecture may be equivalently formulated with (1.1) replaced by:

\[(1.2)\quad a^{-1} e_{hk}(T^0, K^0) \leq e_k(K, T) \leq a e_{k-1}(T^0, K^0)\]

for all \(k \geq 0\) (and there is no loss in generality if we assume that \(k\) is an integer).

As already mentioned, the duality conjecture originated from operator theory, where entropy numbers are used to quantify the compactness of an operator \(u : X \to Y\) between two Banach spaces. Leaving the finite dimensional setting for a brief moment, if \(K = u(B(X))\) and \(T = B(Y)\), where \(B(Z)\) denotes the unit-ball of a Banach space \(Z\), then it is easy to see that \(e_k(K, T) \to 0\) as \(k \to \infty\) iff the operator \(u\) is compact. Since \(u\) is compact iff its adjoint \(u^* : Y^* \to X^*\) is too, and since \(u^*(B(Y^*)) = u^*(T^0)\) and \(B(X^*) = u^*(K^0)\), it follows that \(e_k(K, T) \to 0\) iff \(e_k(T^0, K^0) \to 0\). Hence, it is natural to conjecture that the rate of convergence to 0 is asymptotically similar in both cases. A strong interpretation of this similarity is given by (1.2). We will mention other weaker interpretations below.

Although the general problem is still not completely settled, there has been substantial progress in recent years, and the answer is known to be positive for a wide class of bodies. We begin by describing some results in this direction. We comment here that when the result imposes the same restrictions on \(K\) and \(T\), it is obviously enough to specify only one side of the inequalities in (1.1) or (1.2). When both \(K\) and \(T\) are ellipsoids, it is easy to see that in fact \(N(K, T) = N(T^0, K^0)\). Other special cases were settled in \([19],[5],[6],[9],[12]\). In \([8]\), it was shown that:

\[C^{-n} N(T^0, K^0) \leq N(K, T) \leq C^n N(T^0, K^0),\]

for some universal constant \(C > 1\). This implies that the tail behaviour of the entropy numbers satisfies the duality problem, i.e. \(e_{\lambda k}(K, T) \leq 2e_k(T^0, K^0)\) for some universal constant \(\lambda > 0\) and all \(k \geq n\). This was subsequently generalized in \([17]\).

Another variant of the problem, is to consider not the individual entropy numbers, but rather the entire sequences \(\{e_k(K, T)\}\) and \(\{e_k(T^0, K^0)\}\). Then one may ask whether:

\[(1.3)\quad C^{-1} \|\{e_k(T^0, K^0)\}\| \leq \|\{e_k(K, T)\}\| \leq C \|\{e_k(T^0, K^0)\}\|\]

for some universal constant \(C > 1\) and any symmetric (i.e. invariant to permutations) norm \(\|\cdot\|\). When one of the bodies is an ellipsoid, this was positively settled in \([25]\). Later, in \([4]\), this was extended to the case when one of the bodies is uniformly convex or more generally \(K\)-convex (see \([4]\) and \([18]\) for definitions), in which case the constant \(C\) in (1.3) depends only on the \(K\)-convexity constant. The technique developed in \([4]\) played a crucial role in some of the subsequent results on this problem, and one particular remark will play an essential role in this note.

Returning to the duality problem of individual entropy numbers, it was shown in \([11]\) that there exist universal constants \(a, b \geq 1\) such that when \(T = D\) is an ellipsoid:

\[e_{hk}(D^0, K^0) \leq a(1 + \log k)^3 e_k(K, D),\]

for all \(k \geq 0\). In addition, the authors of \([11]\) observed a connection between (one side of) the duality conjecture with \(T = D\) and a certain geometric lemma. Later, the case when one of the bodies is an ellipsoid was completely settled in \([3]\), by showing that:

\[b^{-1} \log N(D^0, aK^0) \leq \log N(K, D) \leq b \log N(D^0, a^{-1}K^0).\]
The main new tool developed in [3] was the so called “Reduction Lemma”, which roughly reduces the problem (1.1) for all $K, T$ to the case $K \subset 4T$. This will be the second important tool in this note.

Finally, in [2], the Reduction Lemma was combined with the techniques developed in [4], to transfer the results obtained there for the sequence of entropy numbers, to the individual ones. Thus, when one of the bodies $K$ or $T$ is K-convex, (1.1) was shown to hold with the constants $a, b$ depending solely on the K-convexity constant. The key ideological step in [2] was to separate the question of “complexity” from the question of duality, by explicitly introducing a new notion of convexified packing number, which was implicitly used in [4]. We will later refer to this new notion as well.

Our first new observation in this note is in fact an immediate consequence of Theorem 6 in [4] and the Reduction Lemma in [3]. It settles the duality problem (1.1) (and (1.2)) up to $\log(n)$ terms, and in fact strengthens and generalizes all previously known results into a single statement. Because of the symmetry between $K$ and $T$ (as will be explained below), we formulate this as a one sided inequality:

**Theorem 1.1.** Let $K, T$ be two symmetric convex bodies in $\mathbb{R}^n$. Then:

$$
\log N(K, T) \leq V \log(V) \log N(T^\circ, V^{-1}K^\circ),
$$

where $V = \min(V(K), V(T))$ and $V(L)$ is defined as:

$$
V(L) := \inf \{ \log(Cd_{BM}(L, B)) f(K(X_B)); B \text{ is a convex body in } \mathbb{R}^n \},
$$

where $C > 0$ is a universal constant, and $f$ is a function depending solely on $K(X_B)$, the K-convexity constant of the Banach space $X_B$ whose unit ball is $B$.

For more information on the function $f$ see Remark 2.1 below. Recall that the Banach-Mazur distance $d_{BM}(L, B)$ of two symmetric convex bodies $L, B$ is defined as:

$$
d_{BM}(L, B) := \inf \{ \gamma \geq 1; B \subset T(L) \subset \gamma B \},
$$

where the infimum runs over all linear transformations $T$. Since $V(L) = V(L^\circ)$ because $d_{BM}(L, B) = d_{BM}(L^\circ, B^\circ)$ and $K(X_B^\circ) = K(X_B)$, applying the Theorem to $K' = T^\circ$ and $T' = K^\circ$ gives the opposite inequality (with the same $V$):

$$
(V \log(V))^{-1} \log N(T^\circ, VK^\circ) \leq \log N(K, T).
$$

In addition, since by John’s Theorem, the Banach-Mazur distance of any symmetric convex body in $\mathbb{R}^n$ from the Euclidean ball $D$ is at most $\sqrt{n}$, and since $K(D) = 1$, we immediately have:

**Corollary 1.2.** With the same notations as in Theorem 1.1:

$$
\log N(K, T) \leq C \log(1 + n) \log \log(2 + n) \log N(T^\circ, (C\log(1 + n))^{-1}K^\circ),
$$

where $C > 0$ is a universal constant.

This should be compared with the previously known best estimate (to the best of our knowledge) for general symmetric convex bodies $K, T$:

$$
\log N(K, T) \leq C \log N(T^\circ, (Cn)^{-1/2}K^\circ),
$$
which is derived by comparing $K$ with its John ellipsoid and using the duality result of [3]
for ellipsoids. The novelty here in comparison to the results of [2] lies in the logarithmic
dependence in the Banach-Mazur distance.

Although there has been much progress in recent years towards a positive answer to the
duality conjecture, it is still not clear that a positive answer should hold in full generality.
In view of the Corollary 1.2, and Pisier’s well known estimate $\mathcal{K}(X_B) \leq C\log(1+n)$ for
any symmetric convex body $B$ in $\mathbb{R}^n$, we conjecture a weaker form of the duality problem:

**Weak Duality Conjecture for Covering Numbers.** Does there exist a numerical
constant $C \geq 1$ such that for any dimension $n$ and for any two symmetric convex bodies
$K, T$ in $\mathbb{R}^n$ one has:

$$\log N(K, T) \leq V \log N(T^o, V^{-1}K^o),$$

where $V = C \min(\mathcal{K}(X_K), \mathcal{K}(X_T))$?

We present the proof of Theorem 1.1 and several other connections to previously men-
tioned notions in Section 2. In Section 3, we give an application of Corollary 1.2 for
Talagrand’s celebrated $\gamma_p$ functionals, which was in fact our motivation for seeking a result
in the spirit of Corollary 1.2. Recall that for a metric space $(M, d)$ and $p > 0$, $\gamma_p(M, d)$ is
defined as:

$$\gamma_p(M, d) := \inf \sup_{x \in M} \sum_{j \geq 0} 2^{j/p} d(x, M_j)$$

where the infimum runs over all admissible sets $\{M_j\}$, meaning that $M_j \subset M$ and $|M_j| = 2^j$
(we refer to [24, Theorem 1.3.5] and [23] for the connection to equivalent definitions). For
two symmetric convex bodies $K, T$, let us denote $\gamma_p(K, T) := \gamma_p(K, d_T)$, where $d_T$ is the
metric corresponding to the norm induced by $T$. The $\gamma_2(\cdot, D)$ functional, when $D$ is an
ellipsoid, was introduced to study the boundedness of Gaussian processes (see [24] for an
historical account on this topic). It was shown by Talagrand in his celebrated “Majorizing
Measures Theorem” ([21], see also [22],[23]), that in fact $\gamma_2(K, D)$ and $E\sup_{x \in K} \langle x, G \rangle$
where $G$ is a Gaussian r.v. (with covariance corresponding to $D$ in an appropriate manner),
are equivalent to within universal constants. This was later extended to various other classes
of stochastic processes, where the naturally arising metric $d$ is not the $l_2$ norm (again we refer
to [24] for an account).

Our second observation in this note is the following duality relation for the $\gamma_p$ functionals:

**Theorem 1.3.** Let $K, T$ be two symmetric convex bodies in $\mathbb{R}^n$. Then for any $p > 0$:

$$\gamma_p(K, T) \leq C_p \log(1+n)^{2+1/p} \log \log(2+n)^{1/p} \gamma_p(T^o, K^o),$$

where $C_p > 0$ depends solely on $p$.

Although we strongly feel that this is unlikely, one could conjecture that the $\log(n)$ terms
are not required in the last Theorem (at least for some values of $p$). In that case, as will
be evident from the proof, we mention that such a conjecture is independent of the duality
conjecture for covering numbers, in the sense that neither one implies the other.

**Acknowledgments.** I would like to sincerely thank my supervisor Prof. Gideon Schecht-
man for motivating me to prove Proposition 3.3, sharing his knowledge, and for reading
this manuscript.
2. Duality of Entropy

As emphasized in the Introduction, the proof of Theorem 1.1 is immediate once we recall two previously known results. The first is the recently observed “Reduction Lemma” ([3, Proposition 12]), which uses a clever iteration procedure to telescopically expand and reduce the appearing terms. We carefully formulate it below:

Theorem 2.1 ([3]). Let $T$ be a convex symmetric body in a Euclidean space such that, for some constants $a, b \geq 1$, for any convex symmetric body $K \subset 4T$, one has:

$$\log N(K, T) \leq b \log N(T^o, a^{-1}K^o).$$

Then for any convex symmetric body $K$:

$$\log N(K, T) \leq b \log_2(48a) \log N(T^o, (8a)^{-1}K^o).$$

Dually, if $K$ is fixed and the hypothesis holds for all $T$ verifying $K \subset 4T$, then the conclusion holds for any $T$.

The second known result goes back to the work of [4]. It uses the so called Maurey’s Lemma, which (roughly speaking) estimates the covering number of the convex hull of $m$ points by the unit-ball of a $K$-convex space. We combine Theorem 6 and the subsequent remark from [4] into the following:

Theorem 2.2 ([4]). Let $K, T$ be two symmetric convex bodies in $\mathbb{R}^n$, such that $K \subset 4T$. Then:

$$\log N(K, T) \leq V \log N(T^o, V^{-1}K^o),$$

where $V = \min(V(K), V(T))$ and $V(L)$ is given by (1.5).

Combining these two results, we immediately deduce Theorem 1.1. Note that if $V = V(T)$ in Theorem 1.1, we proceed by fixing $T$, applying Theorem 2.2 for all $K$ satisfying $K \subset 4T$ and use the first part of the Reduction Lemma to deduce (1.4) for all $K$; if $V = V(K)$, we fix $K$ and repeat the argument by interchanging the roles of $K$ and $T$ and using the second part of the Reduction Lemma.

Remark 2.1. The proof of Theorem 2.2 in fact gives an explicit expression for $V$, rather than the implicit one used in (1.5):

$$V := C_1 \inf \{\log(C_2\gamma)(10T_p(X_B))^q; K \subset \gamma B, B \subset 4T, \gamma \geq 1\},$$

where the infimum runs over all symmetric convex bodies $B$ in $\mathbb{R}^n$, $T_p(X_B)$ is the type $p$ ($1 < p \leq 2$) constant of the Banach space $X_B$ whose unit-ball is $B$, $q = p^* = p/(p - 1)$ and $C_1, C_2 \geq 1$ are two universal constants (see [10] for the definition of type). Theorem 1.1 was formulated using an implicit function $f$ of $\mathcal{K}(X_B)$, since by several important results of Pisier ([14],[15]), an infinite dimensional Banach Space is $K$-convex iff it has some non-trivial type $p > 1$. We comment that in [15], an explicit formula bounding $\mathcal{K}(X_B)$ as a function of $T_p(X_B)$ and $q$ was obtained. It is possible to obtain an explicit reverse bound using the results in [16], but it is much easier to use an abstract argument which infers the existence of a $p > 1$, depending solely on $\mathcal{K}(X_B)$, such that $T_p(X_B)$ depends solely on $\mathcal{K}(X_B)$ (see, e.g. [7, Lemma 4.2]). The advantage of using the $K$-convexity constant $\mathcal{K}(X_B)$ (instead of $T_p(X_B)$ and $q$), lies in the fact that we may use duality and deduce the other side of the
duality inequality (1.6) with the same $V$, as explained in the Introduction. We also remark that once $V$ in (2.1) is expressed using $K(X_B)$, it is clear that $V \leq \min(V(K), V(T))$ where $V(L)$ is given by (1.5). We need this “separable” estimate on $V$, so that we may apply the Reduction Lemma (where the estimate on one of the bodies must be fixed).

It is important to note that the proof of Theorem 2.2 actually connects the notions of covering and convexified packing, mentioned in the Introduction. For two symmetric convex bodies $K$ and $T$, the convexified packing number, or convex separation number, was defined in [2] as:

$$\hat{M}(K, T) = \max \left\{ N; \exists x_1, \ldots, x_N \in K \text{ such that } (x_j + \text{int } T) \cap \text{conv } \{ x_i; i < j \} = \emptyset \right\}.$$ 

Here int$(T)$ denotes the interior of the set $T$. Note that we always have $\hat{M}(K, T) \leq N(K, T/2)$ by a standard argument (see [2]). Then the proof actually shows:

**Theorem 2.3 ([4]).** Under the same conditions as in Theorem 2.2:

$$\log N(K, T) \leq V \log \hat{M}(K, V^{-1}T).$$

Using John’s Theorem as in Corollary 1.2, we have:

**Corollary 2.4 ([4]).** Let $K, T$ be two symmetric convex bodies in $\mathbb{R}^n$, such that $K \subset 4T$. Then:

$$\log N(K, T) \leq C \log(1 + n) \log \hat{M}(K, (C \log(1 + n))^{-1}T).$$

We mention these variants of Theorem 1.1 and Corollary 1.2 here, because the framework developed in [2] suggests that this is the correct way to understand the duality problem. The cost of transition from covering to convex separation, as given by Theorem 2.3 and Corollary 2.4, is a certain measure of the complexity of the bodies involved. Once the transition is achieved, the duality framework developed in [3] and [2] finishes the job. Indeed, it was shown in [2] that the convex separation numbers always satisfy a duality relation, for any pair of symmetric convex bodies $K, T$:

$$\hat{M}(K, T) \leq \hat{M}(T^o, K^o / 2)^2.$$ 

Using Theorem 2.3, we conclude that when $K \subset 4T$:

$$\log N(K, T) \leq V \log \hat{M}(K, V^{-1}T) \leq 2V \log \hat{M}(T^o, (2V)^{-1}K^o) \leq 2V \log N(T^o, (4V)^{-1}K^o).$$

The Reduction Lemma now immediately gives Theorem 1.1.

To conclude this section, we mention that Theorem 2.3 is already stronger than all of the results in [2] connecting the covering and the convex separation numbers. The technique involving the use of Maurey’s Lemma, which was also used in [2] (see also [1]), is optimally exploited in the proof of Theorem 2.3 (Theorem 6 in [4]), by using a clever iteration procedure, producing the log factor in the various definitions (1.5) and (2.1) of $V$. All previous general results (with no restriction on $K$ and $T$) pay a linear penalty in the Banach-Mazur distance from “low-complexity” bodies, which may be as large as $\sqrt{n}$. 
3. Duality of Talagrand’s $\gamma_p$ Functionals

Given Corollary 1.2, proving Theorem 1.3 is rather elementary, although we will need to collect several elementary observations which we have not been able to find a reference for. Before proceeding, we remark that for our purposes, it is totally immaterial whether the points $\{x_i\}$ in the definition of $N(K,T)$ are chosen to lie inside $K$ or not. Indeed, denoting by $N'(K,T)$ the variant where the points are required to lie inside $K$, it is elementary to check that:

$$N'(K,2T) \leq N(K,T) \leq N'(K,T).$$

Since throughout this note we allow the insertion of homothety constants in all expressions, or multiplying the entropy numbers by universal constants, this lack of distinction is well justified.

First, recall that by Dudley’s entropy bound ([24]):

\begin{equation}
\gamma_p(K,T) \leq C_p \sum_{k \geq 1} k^{1/p-1} e_k(K,T),
\end{equation}

where $C_p > 0$ is some constant depending on $p$. The argument is elementary:

$$\gamma_p(K,T) := \inf \sup_{x \in K} \sum_{j \geq 0} 2^{j/p} d_T(x,K_j) \leq \inf \sum_{j \geq 0} 2^{j/p} \sup_{x \in K} d_T(x,K_j).$$

Choosing $K_j$ to be the set of $2^{2j}$ points (inside $K$) attaining the minimum in the definition of $N(K,e_{2^j}(K,T))$, we see that:

$$\gamma_p(K,T) \leq \sum_{j \geq 0} 2^{j/p} e_{2^j}(K,T).$$

It is elementary to verify that for $p \geq 1$ and $j \geq 1$:

$$2^{j/p} \leq C_p \sum_{k=2^{j-1}}^{2^j-1} k^{1/p-1},$$

where $C_p = (p(1 - 2^{-1/p}))^{-1}$. Since $e_k$ is a non-increasing sequence, we have:

$$\gamma_p(K,T) \leq e_1(K,T) + C_p \sum_{j \geq 1} \sum_{k=2^{j-1}}^{2^j-1} k^{1/p-1} e_{2^j}(K,T) \leq e_1(K,T) + C_p \sum_{k \geq 1} k^{1/p-1} e_k(K,T) \leq (1 + C_p) \sum_{k \geq 1} k^{1/p-1} e_k(K,T).$$

A similar argument works for $0 < p < 1$.

Dudley’s entropy upper bound appears naturally when studying the supremum of Gaussian processes on a set $K$, e.g. $E \sup_{x \in K} \langle x, G \rangle$ where $G$ is a Gaussian r.v. As mentioned in the Introduction, a deep theorem of Talagrand asserts that the latter expectancy is in fact equivalent (to within universal constants) to $\gamma_p(K,D)$ where $D$ is an ellipsoid corresponding to the covariance of $G$. The corresponding lower bound on $E \sup_{x \in K} \langle x, G \rangle$ is due to Sudakov ([20]):

$$E \sup_{x \in K} \langle x, G \rangle \geq c \sup_{k \geq 1} k^{1/2} e_k(K,D).$$
When the body $T$ is not an ellipsoid or when $p \neq 2$, there is no direct connection between Gaussian processes and $\gamma_p(K, T)$. Nevertheless, we note that the analogue to Sudakov’s Minoration bound holds in full generality:

**Lemma 3.1.**

$$\gamma_p(K, T) \geq 2^{-1/p} \sup_{k \geq 1} k^{1/p} e_k(K, T).$$

**Proof.** Let $k \geq 1$ be given, and let $j \geq 0$ be such that $2^j \leq k < 2^{j+1}$. Then:

$$\gamma_p(K, T) := \inf_{x \in K} \sup_{l \geq 0} 2^{l/p} d_T(x, K_l) \geq \inf_{x \in K} 2^{j/p} d_T(x, K_j).$$

Since for any admissible set $K_j$ we have $|K_j| = 2^{2^j} \leq 2^k$, it follows by definition that $\sup_{x \in K} d_T(x, K_j) \geq e_k(K, T)$. Hence:

$$\gamma_p(K, T) \geq 2^{j/p} e_k(K, T) \geq 2^{-1/p} k^{1/p} e_k(K, T).$$

Since $k \geq 1$ was arbitrary, the assertion follows. $\square$

We will need one last lemma for the proof of Theorem 1.3:

**Lemma 3.2.** For all $k \geq 3n$:

$$e_k(K, T) \leq 2e_n(K, T) \exp(-ck/n),$$

where $c > 0$ is some universal constant.

**Proof.** Denote $e_k = e_k(K, T)$ and $e_n = e_n(K, T)$ for short. W.l.o.g. we assume that $N(K, e_k T) = 2^k$ and $N(K, e_n T) = 2^n$. Obviously we have:

(3.2)  $N(K, e_k T) \leq N(K, e_n T)N(e_n T, e_k T).$

Also $N(e_n T, e_k T) = N(T, \frac{e_k}{e_n} T)$, and by a standard volume estimation argument, we can find an $e_k/e_n T$-net of $T$ with cardinality no greater than $(1 + \frac{2e_n}{e_k})^n$. Plugging everything into (3.2), we see that:

$$2^k \leq 2^n \left(1 + \frac{2e_n}{e_k}\right)^n,$$

or equivalently:

$$\exp\left(\log(2) \frac{k - n}{n}\right) - 1 \leq \frac{2e_n}{e_k}.$$

Since $k \geq 3n$, we can find a universal constant $c > 0$ such that:

$$\exp\left(\log(2) \frac{k - n}{n}\right) - 1 \geq \exp\left(c \frac{k}{n}\right).$$

The assertion now readily follows. $\square$

We can now deduce the following equivalence, up to a $\log(n)$ term, of the $\gamma_p$ functional, Sudakov’s lower bound and Dudley’s upper bound. Although this is surely known, we did not find a reference for it, so we include a proof for completeness.
Proposition 3.3. Let $K, T$ denote two symmetric convex bodies in $\mathbb{R}^n$, and denote $e_k = e_k(K, T)$ and $\gamma_p = \gamma_p(K, T)$ for short. Then for any $p > 0$:

$$2^{-1/p} \sup_{k \geq 1} k^{1/p} e_k \leq \gamma_p \leq C_p \sum_{k \geq 1} k^{1/p-1} e_k \leq \log(1 + n)C'_p \sup_{k \geq 1} k^{1/p} e_k,$$

where $C_p, C'_p > 0$ are universal constants depending solely on $p$.

Proof. The first inequality is Sudakov’s lower bound (Lemma 3.1) and the second one is Dudley’s upper bound (3.1). We will show the third inequality. Let us split the sum $\sum_{k \geq 1} k^{1/p-1} e_k$ into two parts, up to and from $k = 3n$. For the first part, we obviously have:

$$\sum_{k=1}^{3n-1} k^{1/p-1} e_k \leq \sum_{k=1}^{3n-1} \frac{1}{k} \sup_{k \geq 1} k^{1/p} e_k \leq C \log(1 + n) \sup_{k \geq 1} k^{1/p} e_k.$$

We use Lemma 3.2 to evaluate the second sum:

$$\sum_{k \geq 3n} k^{1/p-1} e_k \leq 2e_n \sum_{k \geq 3n} k^{1/p-1} \exp(-ck/n).$$

For $p \geq 1$, $k^{1/p-1}$ is non-increasing, so we use:

$$\sum_{k \geq 3n} k^{1/p-1} \exp(-ck/n) \leq (3n)^{1/p-1} \sum_{k \geq 3n} \exp(-ck/n)$$

$$= (3n)^{1/p-1} \frac{\exp(-3c)}{1 - \exp(-c/n)} \leq (3n)^{1/p-1} \exp(-3c) \frac{n}{c} \leq C n^{1/p}.$$

For $0 < p < 1$, we evaluate the sum with an integral (although the series may not be monotone, is has at most one extremal point, and this can be handled by a loose estimate):

$$\sum_{k \geq 3n} k^{1/p-1} \exp(-ck/n) \leq 3 \int_{3n-1}^{\infty} x^{1/p-1} \exp(-cx/n)dx$$

$$\leq 3 \left( \frac{n}{c} \right)^{1/p} \int_{0}^{\infty} x^{1/p-1} \exp(-x)dx = 3c^{-1/p} \Gamma(1/p)n^{1/p}.$$

We conclude that in both cases:

$$\sum_{k \geq 3n} k^{1/p-1} e_k \leq C_p n^{1/p} e_n \leq C'_p \sup_{k \geq 1} k^{1/p} e_k.$$ 

Summing the two parts together, we conclude the proof. \( \square \)

Using Corollary 1.2, the proof of Theorem 1.3 is now clear:

Proof of Theorem 1.3. Corollary 1.2 implies that:

$$e_k(K, T) \leq C \log(1 + n)e_k/(C \log(1 + n) \log \log(2 + n))(T^\circ, K^\circ),$$

for some universal constant $C \geq 1$ and all $k \geq 0$. Using Proposition 3.3 twice, we conclude:

$$\gamma_p(K, T) \leq C_p \log(1 + n) \sup_{k \geq 1} k^{1/p} e_k(K, T)$$

$$\leq C'_p \log(1 + n)^2 (\log(1 + n) \log \log(2 + n))^{1/p} \sup_{k \geq 1} k^{1/p} e_k(T^\circ, K^\circ)$$

$$\leq C_p \log(1 + n)^{2+1/p} \log \log(2 + n)^{1/p} \gamma_p(T^\circ, K^\circ)$$
8. A REMARK ON TWO DUALITY RELATIONS

REFERENCES


E-mail address: emanuel.milman@weizmann.ac.il

DEPARTMENT OF MATHEMATICS, THE WEIZMANN INSTITUTE OF SCIENCE, REHOVOT 76100, ISRAEL.